TITLE Embeddings, Dimension Groups and Presentations of AF Algebras, and the Index of Subfactors.

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Embeddings, Dimension Groups and Presentations of AF Algebras, and the Index of Subfactors.

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Summary:

If Γ is a graph, with distinguished vertex $*$, let $A(\Gamma)$ denote the path algebra on the space $\hat{\Gamma}$ of semi-infinite paths in Γ beginning at $*$. We discuss embeddings $A(\Gamma_1) \to A(\Gamma_2)$ of AF algebras associated with graphs Γ_1 and Γ_2 from a dimension group point of view. For certain infinite T-shaped graphs, we have $K_0(A(\Gamma)) \cong \mathbb{Z}[t]$, with positive cone identified with $\{0\} \cup \{P \in \mathbb{Z}[t] : P(\lambda)>0, \lambda \in (0,\gamma]\}$, where $\gamma = \gamma(\Gamma) = ||\Gamma||^{-2} < \frac{1}{4}$. Hence for certain graphs there exists a unital homomorphism $A(\Gamma_1) \to A(\Gamma_2)$ if $||\Gamma_1|| \leq ||\Gamma_2||$. For certain finite T-shaped graphs $K_0(A(\Gamma)) \cong \mathbb{Z}[t]/\langle Q\rangle$ where $\langle Q\rangle$ denotes the ideal generated by a polynomial $Q = Q(\Gamma)$ which is essentially the characteristic polynomial of the graph Γ, and positive cone identified with $\{0\} \cup \{f + \langle Q\rangle : f(\gamma) > 0\}$ where $\gamma = \gamma(\Gamma) = ||\Gamma||^{-2}$. Hence there exists a unital homomorphism $A(\Gamma_1) \to A(\Gamma_2)$ if $||\Gamma_1|| = ||\Gamma_2||$, and $Q(\Gamma_2)$ divides $Q(\Gamma_1)$. If Γ is connected, and locally finite then we construct embeddings of $A(A_m)$ in $A(\Gamma)$ for suitable m, $3 \leq m \leq \infty$, depending on $||\Gamma||$, by giving explicit matrices of multiplicities for the embeddings of the finite dimensional subalgebras of $A(A_m)$ in those of $A(\Gamma)$. These matrices are non-negative integer intertwiners of the incidence matrices of the graphs A_m and Γ. Taking von Neumann algebra completions of these embeddings gives, in certain cases, irreducible subfactors of the hyperfinite type II_1 factor. The structure of $K_0(A(\Gamma))$ as an ordered ring is related to the fusion rules of rational conformal field theory. Moreover for these T-shaped graphs we give an algebraic presentation similar to that for the Temperley-Lieb-Jones algebra which further illuminates the above embeddings. This presentation involves a new projection in addition to those of Temperley-Lieb-Jones. This gives us a rigidity above index four.

Table of Contents.

List of Figures.

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Declaration.

Section 2 contains preliminary material and is not new. Except where stated otherwise, the material in section 3 onwards, to the end of the thesis, is original. Most of it is based on joint research with my supervisor Professor D.E. Evans, and will be submitted for publication [EG], although throughout most of the details have been worked out by myself.

§1. Introduction.

The problem of classifying subfactors of type II_1 factors was initiated by V. Jones [J] with the introduction of a conjugacy invariant called the index. If M is a type II_1 factor and N a subfactor, then the index of N in M is the dimension of M considered as a module over N. To determine the range of values of the index, Jones introduced a family of projections $\{e_i : i \in \mathbf{N}\}$, canonically associated with the pair $N \subseteq M$ when the index $[M:N]$ is finite, and satisfying the relations

(1.1) $$e_n e_{n\pm 1} e_n = \tau e_n, \quad e_n e_m = e_m e_n, \quad |m-n| > 1,$$

(1.2) $$\operatorname{Tr}(x e_m) = \tau \operatorname{Tr}(x), \quad x \in C^*(1, e_1, \ldots, e_{m-1}),$$

where τ is the reciprocal of the index, and Tr is the trace on M. Let $A(\tau)$ be the C^*-algebra generated by a unit, and projections $\{e_i : i \in \mathbf{N}\}$, satisfying relations (1.1) and (1.2) for some $\tau \in \mathbf{R}$, and trace Tr on $A(\tau)$. Jones proved that $A(\tau)$ is non-trivial if and only if

(1.3) $$\tau^{-1} \in \{4\cos^2(\pi/\ell) : \ell = 3,4,\ldots\} \cup [4,\infty),$$

and thus that the index $[M:N]$ is contained in the above set, or is infinite. He also showed that the von Neumann algebra generated by the projections $\{e_i : i \in \mathbf{N}\}$, with parameter τ, is the hyperfinite type II_1 factor, and that the von Neumann algebra generated by the projections: $e_2, e_3, e_4, \ldots$, is a subfactor with index τ^{-1}. Thus there are subfactors of the hyperfinite type II_1 factor realizing each of the values in (1.3).

Jones showed that $A(\tau)$ is an AF algebra whose structure depends on the parameter τ, and may be described as follows. Let Γ be a connected, locally finite graph, with distinguished vertex $*$, let $A(\Gamma)$ denote the path algebra on the space $\hat{\Gamma}$ of semi-infinite paths in Γ beginning at $*$. Let $\Gamma^{(0)}$ and $\Gamma^{(1)}$ denote the vertices and edges respectively of Γ, and Δ the incidence matrix of Γ. We write $\|\Gamma\| = \|\Delta\|$. A Markov trace Tr on $A(\Gamma)$ is given by a solution $(\phi_v : v \in \Gamma^{(0)}) > 0$ to

(1.4) $$x\phi_v = \sum_{w\in\Gamma^{(o)}} \Delta(v,w)\phi_w .$$

Then $A(\tau)$ is described by

(1.5) $$A(\tau) \cong A(A_\ell) \quad \text{if } \tau^{-1} = 4\cos^2(\pi/\ell), \quad \ell = 3,4,...$$

(1.6) $$A(\tau) \cong A(A_\infty) \quad \text{if } \tau^{-1} \geq 4,$$

where A_m , $3 \leq m \leq \infty$, denote the usual Dynkin diagrams. In general one can define canonical projections $\{e_i : i \in N\}$ in $A(\Gamma)$ which satisfy the relations (1.1), and (1.2) with respect to the Markov trace Tr ,where $\tau^{-1} = ||\Gamma||^2$ [O,S]. Thus we have an inclusion of a pair of AF algebras

(1.7) $$A(A_m) \cong A(\tau) \subseteq A(\Gamma).$$

Jones showed that subfactors of index less than four are irreducible, i.e. $N' \cap M = \mathbb{C}1$, but it is not known which values greater than four are allowable for irreducible subfactors. Examples of such subfactors were constructed in [GHJ] by taking von Neumann algebra completions of the embeddings $A(\tau) \subseteq A(\Gamma)$ in (1.7), with Γ a Dynkin diagram of type A-D-E, and the distinguished vertex $\bullet$ varying over the vertices of Γ. The minimum value, greater than four they obtained for an irreducible subfactor being $3 + \sqrt{3}$ when $\Gamma = E_6$.

Subfactors of the hyperfinite type II_1 factor, with index less than four have been classified by Ocneanu [O]. He showed that such subfactors may be obtained as von Neumann algebra completions of certain embeddings $A(\Gamma) \subseteq A(\Gamma)$, of AF algebras, where Γ is a Dynkin diagram of type A_n, $n = 3,4,...$, D_{2n}, $n = 2,3,...$, E_6, E_8. The subfactors constructed by Jones correspond to the A_m series, although his construction did not explicitly use the fact that $A(\tau)$ was an AF algebra.

The AF algebras $A(\Gamma)$, and the family of projections $\{e_i : i \in N\}$ satisfying (1.1), (1.2) also occur in the transfer matrix method for certain classical statistical mechanical models on two dimensional lattices [TL,P1,KAW]. Let Γ be a graph, then one can consider a classical model on a two dimensional square lattice L, whose configurations consist of distributions, $\sigma \in (\Gamma^{(0)})^L$, of vertices $\Gamma^{(0)}$ on the lattice L, such that the vertices σ_a, σ_b, for nearest neighbours a, b in L are joined by an edge in $\Gamma^{(1)}$, and a Hamiltonian with interactions around faces. Thus if Γ is the Dynkin diagram A_m one recovers the models of Andrews, Baxter and Forrester [ABF]. For a particular choice of Bolzman weights, the algebra of the transfer matrices is described by the projections $\{e_i : i \in N\}$, in $A(\Gamma)$.

In the following we discuss embeddings $A(\Gamma_1) \subseteq A(\Gamma_2)$ of AF algebras associated with graphs Γ_1 and Γ_2 from a dimension group point of view, and by obtaining algebraic presentations for the algebras $A(\Gamma)$ for certain graphs Γ.

To be more specific, in §7 we give an algebraic characterization of $A(T_{p,2,r})$ for $1 \leq r \leq \infty$ (see Figures 5, 6). Let $e_{\bar{p}}, e_1, e_2, \ldots$ be a sequence of projections satisfying relations (1.1) and additionally:

(1.8) $\quad e_{\bar{p}} e_n = e_n e_{\bar{p}} \qquad n = 1,2, \ldots, p-1, p+1, p+2, \ldots$

(1.9) $\quad e_{\bar{p}} e_p e_{\bar{p}} = \tau e_{\bar{p}}$

(1.10) $\quad e_p e_{\bar{p}} e_p = \tau (1 - e_1 \vee \ldots \vee e_{p-2}) e_p.$

Then we show in Theorem 8.1 that $A(\tau,p) = C^*(1, e_{\bar{p}}, e_1, e_2, \ldots)$ is non-trivial only when

$$\beta = \tau^{-\frac{1}{2}} \in \{\|T_{p,2,r}\| : r \geq 1\} \cup [\|T_{p,2,\infty}\|, \infty) \qquad (1.11).$$

In which case there exists a surjective *-homomorphism

(1.12) $\quad \psi : A(T_{p,2,r}) \oplus \mathbb{C}(1 - e_1 \vee \ldots \vee e_{p+r-2} \vee e_{\bar{p}}) \to A(\tau,p),$

when $\beta = \|T_{p,2,r}\|, r < \infty.$

(1.13) $\quad A(T_{p,2,\infty}) \to A(\tau,p)$, when $\beta \geq \|T_{p,2,\infty}\|$.

If $r < \infty$, i.e. $\beta < \|T_{p,2,\infty}\|$, then this map (1.12) is automatically an isomorphism as $A(T_{p,2,r})$ is simple. If there exists a Markov trace on $A(\tau,p)$ (c.f. (1.2), or see §2 and the statement of Theorem 8.1 for a precise definition) then in all cases (1.12) and (1.13) we have an isomorphism between $A(T_{p,2,r})$, $1 \le r \le \infty$ and $A(\tau,p)$; moreover in the case $r < \infty$, $\beta = \|T_{p,2,r}\|$, we have

$$(1.14) \qquad 1 = e_1 \vee \dots \vee e_{p+r-2} \vee e_{\hat{p}} \, .$$

We give a constructive proof of the existence of the above homomorphisms (1.12)-(1.13) constructing matrix units in $A(\tau,p)$ labelled by paths in the graph $\hat{T}_{p,2,r}$. Thus even in the case of $p = 1$, our proof does not reduce to that of Jones for the A_n-series. Indeed we prove a stronger result in that the existence of the homomorphism in (1.12) and (1.13) does not depend on the existence of a Markov trace. Moreover we show that the homomorphism in (1.12) is an isomorphism even without the assumption of a Markov trace. It is also striking to note that by throwing in the extra relations (1.8)-(1.10) to those of Temperley-Lieb and Jones (1.1), we find a rigidity *above* index four. Note also that our construction of matrix units is different from that proposed by [T] in the A_n-case.

Thus if Γ is a graph, we see from the preceding that for suitable ℓ $(3 \le \ell \le \infty)$, $A(A_\ell)$ is embedded in $A(\Gamma)$. We wish to find *explicit* embeddings, in the sense of how to map the finite dimensional subalgebras of $A(A_\ell)$ in those of $A(\Gamma)$, with *explicit* matrices of multiplicities. If Γ is a connected locally finite, bipartite graph, with $\|\Gamma\| = 2\cos \pi/\ell$, then an *explicit* embedding of $A(A_\ell)$ in $A(\Gamma)$, is constructed in §4, whilst if $\|\Gamma\| \ge 2$, an *explicit* embedding of $A(A_\infty)$ in $A(\Gamma)$ is constructed. Embeddings of $A(\Gamma_1)$ in $A(\Gamma_2)$ for graphs Γ_1 and Γ_2 is related to the problem of finding an intertwining matrix for the incidence matrices of Γ_1 and Γ_2 which transforms the distinguished vertex of Γ_1 into that for Γ_2. Such problems are familiar in the study of topological Markov chains, where the theory of dimension groups has proved useful. So to understand the above embeddings better, we study in §3, the dimension groups of $A(T_{p,q,\infty})$ where $T_{p,q,\infty}$ is the graph with the vertices labelled as shown in Figure 1. Thus $T_{2,1,\infty} = A_\infty$, $T_{2,2,\infty} = D_\infty$, $T_{3,2,\infty} = E_\infty$, as in Figure 2.

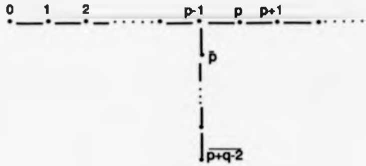

Figure 1 : $T_{p,q,\infty}$

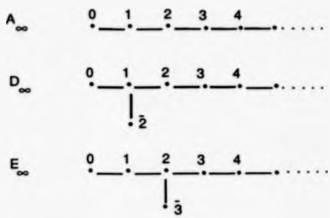

Figure 2

It is known [BP] that $A(A_\infty) \cong (\bigotimes_1^\infty M_2)^{SU(2)}$, and that [W] $K_0(A(A_\infty)) \cong \mathbb{Z}[t]$, where the positive cone can be identified with

$$\{0\} \cup \{Q \in \mathbb{Z}[t] : Q(\lambda) > 0,\ \ \lambda \in (0,\tfrac{1}{4}]\}.$$

We show in §3 that $K_0(A(T_{p,2,\infty})) \cong \mathbb{Z}[t]$, where the positive cone is identified with

$$\{0\} \cup \{Q \in \mathbb{Z}[t] : Q(\lambda) > 0,\ \ \lambda \in (0,\gamma_p]\}$$

where $\gamma_p = \|T_{p,2,\infty}\|^{-2} \le \frac{1}{4}$ is decreasing in p. That these are indeed dimension groups is a consequence of the work of Bratteli, Elliott and Herman [BEH] but no general construction of the AF algebras associated to such dimension groups was known until now.

For a graph Γ, we seek rational functions $\{\phi_v : v \in \Gamma^{(0)}\}$ in an indeterminate x such that

$$x\phi_v = \sum_{w \in \Gamma^{(0)}} \Delta(v,w)\phi_w$$

$$\phi_* = 1.$$

If $d(v)$ is the distance of a minimal path from $*$ to v, let

$$Q_v(t) = x^{-d(v)}\phi_v(x)$$

where $t = x^{-2}$. For the graphs $T_{p,2,\infty}$, the polynomials

$$\{t^m Q_v(t) : m = 0,1,2,\dots,\ v \in \Gamma^{(0)}\},$$

conveniently relabelled as $\{Q_\alpha(t) : \alpha \in \hat{\Gamma}^{(0)}\}$ will generate the positive cone $K_0(A(T_{2,p,\infty}))_+$ in $K_0(A(T_{p,2,\infty})) \cong \mathbb{Z}[t]$. In particular we have a positive map $K_0(A(A_\infty)) \to K_0(A(T_{p,2,\infty}))$. That this map arises from a unital $*$-homomorphism $A(A_\infty) \to A(T_{p,2,\infty})$ follows from Elliott's theorem [Ell], but the problem is to find an explicit lifting. An explicit lifting of this map to the algebra level amounts to being able to write polynomials which generate $\{Q \in \mathbb{Z}[t] : Q(\lambda) > 0,\ \ \lambda \in (0,\frac{1}{4}]\}$, essentially Chebyshev polynomials of the second kind, as a positive linear combination of polynomials which generate $\{Q \in \mathbb{Z}[t] : Q(\lambda) > 0,\ \ \lambda \in (0,\gamma_p]\}$. Explicit embeddings at the algebra level are constructed in section 4 with the aid of some work of Hamachi [H] on the Jones index in an ergodic theory context. These embeddings are best understood in terms of the polynomials $\{\phi_v(x) : v \in \Gamma^{(0)}\}$ rather than $\{Q_\alpha : \alpha \in \hat{\Gamma}^{(0)}\}$. In particular by careful pruning of the map from $A(A_\infty)$ into $A(E_\infty)$ we can give an explicit embedding of $A(A_{11})$ in $A(E_6)$. This embedding was also described by [P2] using ad hoc methods. Taking suitable von Neumann algebra completions, one obtains in §7 the examples of Goodman, de la Harpe and Jones [GHJ] of irreducible subfactors of the hyperfinite II_1 factor with index greater than four, in particular the example with index $3+\sqrt{3}$. (Note that for suitable p, and r the von Neumann algebra generated by projections: $e_{\bar{p}}, e_1, e_2, \ldots$, satisfying (1.1), (1.8)-(1.10) is the hyperfinite type II_1 factor, and the projections: $e_1, e_2, \ldots$, generate a subfactor. This gives another realization of these examples.) In fact for $p \geq q$, $q = 1,2$, $r < \infty$ we show in §5 that $K_0(A(T_{p,q,r})) \cong \mathbb{Z}[t] / \langle Q\rangle$ where $\langle Q\rangle$ is the ideal in $\mathbb{Z}[t]$ generated by the polynomial Q which is essentially the characteristic polynomial of $T_{p,q,r}$, and that the positive cone corresponds to $\{0\} \cup \{f + \langle Q\rangle : f(\gamma) > 0\}$, where $\gamma = ||T_{p,q,r}||^{-2}$.

The ordered ring structure of $K_0(A(\Gamma))$ is related to the fusion rules of rational conformal field theory [V,MS1,MS2]. A two-dimensional lattice model gives rise to a one-dimensional quantum lattice model via the transfer matrix method. At criticality the quantum model gives rise to a field theory in two dimensions, which is scale invariant - a conformal field theory. In a rational conformal field theory, the physical Hilbert space $\mathcal{H}$ decomposes as a finite sum

$$\mathcal{H} = \bigoplus_{r,\bar{r}=0}^{N} h_{r,\bar{r}} \; \mathcal{H}_r \otimes \mathcal{H}_{\bar{r}} \tag{1.15}$$

where $\mathcal{H}_r$ is an irreducible representation of the chiral algebra $\mathcal{A}$, and $h_{r,\bar{r}}$ represents a multiplicity . Chiral vertex operators are intertwining operators. Given three representations i,j,k, then the chiral vertex operators of type $\binom{i}{j\,k}$ will be a vector space V^i_{jk} of linear transformations $\mathcal{H}_i^\vee \otimes \mathcal{H}_j \otimes \mathcal{H}_k \to \mathbb{C}$ The fusion rules are $N^i_{jk} = \dim V^i_{jk}$. With primary fields $\{\phi_i\}$ we have the formal product rule or operator product expansion:

$$\phi_j \times \phi_k = \sum N^i_{jk}\, \phi_i. \tag{1.16}$$

For certain graphs Γ considered in section 3 and section 4, $K_o(A(\Gamma))$ will be an ordered ring, and $K_o(A(\Gamma))_+$ has generators $\{Q_\alpha : \alpha \in \hat{\Gamma}^{(0)}\}$. So from the ordered ring structure we have non-negative integers $a^{\gamma}_{\alpha\beta}$, $\alpha,\beta,\gamma \in \hat{\Gamma}^{(0)}$, such that

$$Q_\alpha Q_\beta = \sum a^{\gamma}_{\alpha\beta} Q_\gamma \tag{1.17}$$

In § 6 we discuss in more detail the fusion rules of (1.16) (respectively (1.17)) for certain models (respectively graphs) e.g. Wess-Zumino-Witten $SU(2)_k$ models (respectively A_ℓ). Again, it seems that (1.4) is best understood in terms of the set of polynomials $\{\phi_v(x)\}$ indexed by the vertices v of the graph Γ, rather than the set $\{Q_\alpha(t)\}$ indexed by the vertices α of the Bratteli diagram $\hat{\Gamma}$.

§2. Preliminaries.

AF algebras.

Let M be a finite dimensional C^*-algebra. Then M is semisimple, and so has a decomposition

$$M = \bigoplus_{j=1}^{n} M_j$$

where M_j is isomorphic to the algebra $M_{k_j}(\mathbb{C})$, of $k_j \times k_j$ matrices over the complex numbers for $j = 1,\ldots,n$. The sequence of positive integers $k = (k_1,\ldots,k_n)^T$, that occurs in this decomposition is unique up to a permutation. On the other hand, given any finite sequence of positive integers $k = (k_1,\ldots,k_n)^T$, one defines a multimatrix algebra

$$M(k) = \bigoplus_{j=1}^{n} M_{k_j}(\mathbb{C}) \qquad (2.1).$$

It is clear that $M(k)$ is a finite dimensional C^*-algebra. In the following we will not distinguish between finite dimensional C^*- algebras and multimatrix algebras.

Homomorphisms of multimatrix algebras may be defined as follows. Let $N = M(k)$, and $M = M(l)$ with $k \in \mathbb{N}^n, l \in \mathbb{N}^m$. Let $A = (a_{i,j})$ be an $m \times n$ non-negative integer matrix with no columns zero such that

$$A k \leq l \qquad (2.2),$$

i.e. each entry of l exceeds the corresponding entry of Ak. Then if $x = x_1 \oplus x_2 \oplus \ldots \oplus x_n$, with $x_j \in M_{k_j}(\mathbb{C})$, for $j = 1,\ldots,n$, define $\phi: N \to M$ by

$$\phi(x) = y_1 \oplus y_2 \oplus \ldots \oplus y_m$$

where $y_i \in M_{l_i}(\mathbb{C})$, $i = 1,\ldots,m$, is given by

$$y_i = x_1 \oplus x_1 \oplus \ldots \oplus x_1 \oplus x_2 \oplus \ldots \oplus x_2 \oplus \ldots \oplus x_n \oplus \ldots \oplus x_n$$

with x_j appearing $a_{i,j}$ times along the diagonal of y_i. A homomorphism of this type is called canonical. Note that ϕ is injective, and if $Ak = l$ then ϕ is unital, i.e. $\phi(1_N) = 1_M$, where 1_N, and 1_M denote the units of N, and M respectively. Now since any non-trivial representation of a simple matrix algebra is equivalent to a direct sum of identity representations, it follows that every homomorphism $\varphi: N \to M$, of multimatrix algebras is inner equivalent to a unique canonical homomorphism. This means that there exists a unitary u in M such that ϕ defined by

$$\phi(x) = u^*\varphi(x)u ,$$

for $x \in N$, is canonical. The matrix A, satisfying (2.2), determined by φ is called the inclusion matrix. Bratteli [B] introduced a graph Γ to describe a homomorphism $\varphi: N \to M$. The vertices of Γ correspond to the simple components of N, and M. If the homomorphism is given by the inclusion matrix $A = (a_{i,j})$, then there are $a_{i,j}$ edges between vertices i and j, where i and j correspond to simple components of M, and N respectively. The incidence matrix for Γ is

$$\Delta = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}.$$

A positive trace, Tr, on a multimatrix algebra $N = M(k)$ is specified by a vector $s = (s_1,\dots,s_n)^T$, where s_i is a positive real number giving the trace of a minimal idempotent in the i^{th} simple component of N for $i = 1,\dots,n$. The trace will be normalized, i.e. $Tr(1_N) = 1$, if $s^T k = 1$. If there is a unital inclusion of the multimatrix algebra $M(k)$ into $M(l)$ given by the inclusion matrix A, and the trace on $M(l)$ is specified by t, then the restriction of the trace to $M(k)$ will be given by $s = A^T t$.

Let B_k be finite dimensional C*-algebras for $k = 1,2,\dots$, and suppose there is a system of homomorphisms

$$B_1 \xrightarrow{\phi_1} B_2 \xrightarrow{\phi_2} B_3 \xrightarrow{\phi_3} \dots \qquad (2.3).$$

Then one can form the direct limit, $B_\infty = \varinjlim B_k$. This inherits a natural *-algebra structure from the B_k's, and has a unique norm because of the compatibility of the norms on the B_k's. The completion of B_∞ with respect to this norm is a C*-algebra, which we denote by $\overline{\cup B_k}$. Any C*-algebra with a dense *-subalgebra, which is the direct limit of an ascending sequence of finite dimensional subalgebras, is called an AF algebra [B]. In the following we will only consider unital AF algebras. These occur as direct limits of systems (2.3) for which the connecting homomorphisms, $\phi_k: B_k \to B_{k+1}$, $k = 1,2,\dots$, are unital. By concatenating the graphs for these homomorphisms one obtains an infinite graph known as the Bratteli diagram [B]. There are certain axioms characterizing Bratteli diagrams, and the diagrams determine AF algebras up to isomorphism. Note that a given AF algebra can be described by more than one diagram.

Let Γ be a graph with distinguished vertex $*$, and denote by $\Gamma^{(0)}$, and $\Gamma^{(1)}$, the vertices, and edges respectively. Let Δ denote the incidence matrix for Γ. We assume throughout that Γ is connected and locally finite, i.e. the number of edges adjacent to a vertex is finite, and also that there is at most one edge joining any two vertices. We say that $v \in \Gamma^{(0)}$ is even (respectively odd) if it can be joined to $*$ by an even (respectively odd) number of vertices.

We now describe a version of the the path model for AF algebras [CE,Eva1,Eva2,GHJ]. Let $\hat{\Gamma}$ denote the Cantor set of sequences $\gamma = (v_k)_{k=0}^{\infty}$, with $(v_k, v_{k+1}) \in \Gamma^{(1)}$, $v_0 = *$, and put $\gamma(k) = v_k$. For $\gamma \in \hat{\Gamma}$, and $0 \le r \le s \le \infty$, we define $\gamma_{[r,s]} = (v_k)_{k=r}^{s}$, and put

$$\hat{\Gamma}_{[r,s]} = \{ \gamma_{[r,s]} ; \gamma \in \hat{\Gamma} \}.$$

Except where it may cause confusion, we write $\gamma = \gamma_{[r,s]}$. For $\gamma = \gamma_{[r,s]} \in \hat{\Gamma}_{[r,s]}$, and $\xi = \xi_{[s,t]} \in \hat{\Gamma}_{[s,t]}$, with $\gamma(s) = \xi(s)$, define $\gamma \circ \xi \in \hat{\Gamma}_{[r,t]}$ by $\gamma \circ \xi(k) = \gamma(k)$, if $r \le k \le s$, and $\gamma \circ \xi(k) = \xi(k)$, if $s \le k \le t$. Now put

$$G_{[r,s]} = \Big\{ (\eta,\xi) \in \hat{\Gamma}_{[r,s]} \times \hat{\Gamma}_{[r,s]} ; \eta(r) = \xi(r), \eta(s) = \xi(s) \Big\},$$

and for $(\eta,\xi) \in G_{[r,s]}$, define $f_{\eta,\xi} \in \mathrm{End}\,(\ell^2(\hat{\Gamma}))$ by

$$f_{\eta,\xi}(\gamma) = \delta(\xi_{[r,s]}, \gamma_{[r,s]})\, \gamma_{[0,r]} \circ \eta_{[r,s]} \circ \gamma_{[s,\infty]},$$

for $\gamma \in \hat{\Gamma}$. Then for $(\eta,\xi), (\eta',\xi') \in G_{[r,s]}$, one has

$$f_{\eta,\xi}\, f_{\eta',\xi'} = \delta(\xi,\eta')\, f_{\eta,\xi'} \tag{2.4}$$

and so if we let

$$A(\Gamma)_{[r,s]} = \mathrm{lin}_{\mathbb{C}} \{ f_{\eta,\xi} ; (\eta,\xi) \in G_{[r,s]} \},$$

then in view of (2.4) $A(\Gamma)_{[r,s]}$ is an associative algebra with unit:

$$1 = \sum_{\gamma \in \hat{\Gamma}_{[rs]}} f_{\gamma\gamma}.$$

In fact $A(\Gamma)_{[r,s]}$ is semisimple, and has the decomposition:

$$A(\Gamma)_{[r,s]} = \bigoplus_{\gamma \in \hat{\Gamma}_{[rr]}\, \xi \in \hat{\Gamma}_{[ss]}} M_{(\gamma(r),\xi(s))} \tag{2.5}$$

where the simple components are given by

$$M_{(v,w)} = \mathrm{lin}_{\mathbb{C}} \{ f_{\eta,\xi} ; \eta,\xi \in t_{[r,s]}(v,w) \} \cong \mathrm{End}\, (\ell^2(t_{[r,s]}(v,w))),$$

with

$$t_{[r,s]}(v,w) = \{ \gamma \in \hat{\Gamma}_{[r,s]} ; \gamma(r) = v,\ \gamma(s) = w \}.$$

Now put $A(\Gamma)_n = A(\Gamma)_{[0,n]}$, then there is a unital inclusion of $A(\Gamma)_n$ into $A(\Gamma)_{n+1}$ since

$$f_{\eta,\xi} = \sum f_{\eta\circ\lambda\ \xi\circ\lambda},$$

where the summation is over all $\lambda \in \hat{\Gamma}_{[n,n+1]}$ with $\lambda(n) = \eta(n)$. The system of finite dimensional C*-algebras

$$A(\Gamma)_0 \to A(\Gamma)_1 \to A(\Gamma)_2 \to \dots,$$

determines an AF algebra, $A(\Gamma)$, whose Bratteli diagram may be identified with $\hat{\Gamma}$.

Note that there is a unital inclusion of $A(\Gamma)_{[r,s]}$ into $A(\Gamma)_s$ since for $(\eta,\xi) \in G_{[r,s]}$ we have

$$f_{\eta,\xi} = \sum f_{\lambda\circ\eta\ \lambda\circ\xi},$$

where the summation is over all $\lambda \in \hat{\Gamma}_{[0,r]}$ with $\lambda(r) = \eta(r)$, and it can be shown that $A(\Gamma)_{[r,s]} = A(\Gamma)_r' \cap A(\Gamma)_s$.

We now define a sequence of projections $e_n \in A(\Gamma)_{n-1}' \cap A(\Gamma)_{n+1}$, for $n = 1,2,\ldots$. First note that we can identify the set $t_{[n-1,n+1]}(v,v)$, with the set $t(v) = \{w \in \Gamma^{(0)} : (v,w) \in \Gamma^{(1)}\}$. Suppose that $(\phi_v : v \in \Gamma^{(0)}) > 0$ is a solution to

$$x\phi_v = \sum_{w \in \Gamma^{(0)}} \Delta(v,w)\phi_w \qquad (2.6).$$

Then $X(v) = ((\phi_w/\phi_v x)^{\frac{1}{2}} : w \in t(v))$ defines a unit vector in $\ell^2(t(v))$. Then we define e_n, to be the projection, whose restriction to $\mathrm{End}(\ell^2(t(v)))$, in (2.5) is the rank one projections onto $X(v)$, for each $v \in \hat{\Gamma}_{[n-1,n+1]}$.

The family $\{e_n : n = 1,2,3,\ldots\}$ satisfy the relations

$$e_n e_{n\pm 1} e_n = \tau e_n, \quad e_n e_m = e_m e_n, \ |m - n| \geq 2 \qquad (2.7).$$

where $\tau = x^{-2}$ [O,S], and are called the canonical projections in $A(\Gamma)$.

We define a trace Tr, called a Markov trace, on $A(\Gamma)$ to be the unique state on $A(\Gamma)$ such that

$$\mathrm{Tr}\, f_{\gamma,\gamma'} = 0 \quad \text{if} \quad \gamma \neq \gamma' \quad \gamma, \gamma' \in \hat{\Gamma}_{[0,n]}$$

$$\mathrm{Tr}\, f_{\gamma,\gamma} = x^{r-s}\phi_v\phi_w \quad \text{if} \quad \gamma \in t_{[r,s]}(v,w)$$

Then

$$\mathrm{Tr}\,(y\, e_m) = x^{-2}\, \mathrm{Tr}\,(y) \qquad y \in A(\Gamma)_m$$

$$\mathrm{Tr}\,(e_m) = x^{-2}, \quad \mathrm{Tr}\,(1) = 1.$$

Note that if the graph Γ is finite and connected, then by the Perron Frobenius theory there is an unique normalized strictly positive solution to (2.6) and $x = \|\Delta\|$.

If Γ contains no cycle of odd length, we say the graph is bipartite, and $\Gamma_+^{(o)} \cap \Gamma_-^{(o)} = \emptyset$. Then it is more convenient to describe $\hat{\Gamma}$ as follows [O]. There is a distance function $d : \Gamma^{(o)} \to N$, where $d(v)$ is the number of edges in a minimal path from $*$ to v. Then we can identify

$$\hat{\Gamma}^{(o)} = \{(v,d(v) + 2k) : v \in \Gamma^{(o)},\ k = 0,1,2,\ldots\}$$

with distinguished vertex $(*,o)$, and where there are p edges between vertices (v,n) and (w,m) in $\hat{\Gamma}^{(o)}$ if and only if $|n - m| = 1$ and there are p edges between v and w in $\Gamma^{(o)}$. We identify Γ with the subgraph of $\hat{\Gamma}$, called the underlying graph, having vertices

$$\{(v,d(v)) : v \in \Gamma^{(o)}\}$$

and whose edges are those in $\hat{\Gamma}^{(o)}$ connecting these vertices. The distance function d on Γ extends to a distance function on $\hat{\Gamma}^{(o)}$ also denoted by d, where $d(v,m) = m$.

Dimension groups.

An ordered group is an abelian group G, with a distinguished subset G_+, called the positive cone such that:

(i) G is partially ordered as a group, i.e. $G_+ + G_+ \subseteq G_+$, and $G_+ \cap (-G_+) = \{0\}$,

(ii) G is directed, i.e. $G_+ - G_+ = G$,

(iii) G is unperforated, i.e. if $a \in G$, and $n \in N$, then $na \in G_+$, implies $a \in G_+$

If $a - b \in G_+$, then we write $a \geq b$.

Let $G = \mathbb{Z}^n$, and put $G_+ = \{ (a_1,\ldots,a_n) \in \mathbb{Z}^n;\ a_i \geq o \}$, then G is an ordered group. The positive cone, G_+, is called the simplicial ordering. Let G, and H be ordered groups, and $f: G \to H$ a group homomorphism, then f is said to be positive if f maps the positive cone of G into the positive cone of H, i.e. $f(G_+) \subseteq H_+$. An order isomorphism $f: G \to H$ is a group isomorphism with both f, and f^{-1} positive. An ordered group that is order isomorphic to $\mathbb{Z}^n$ with the simplicial ordering is called a simplicial group.

Consider the system of ordered groups and positive homomorphisms

$$G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \xrightarrow{f_3} \dots \ .$$

The direct limit $G_\infty = \varinjlim G_n$, equipped with the inherited group structure, and positive cone, is an ordered group. An ordered group arising as the direct limit of a system of simplicial group, is called a dimension group. Dimension groups have an intrinsic characterization [EHS] as being those countable ordered groups, G, which possess the Riesz interpolation property, i.e. if a_1, a_2, b_1, b_2, are elements of G such that $a_i \le b_j$, for i,j = 1,2, then there is an element c in G such that $a_i \le c \le b_j$, for i,j = 1,2.

Let G be an ordered group then an element u of G is called an order unit if for every positive g in G, there is a positive integer n, such that $g \le nu$. A positive homomorphism $\phi: G \to \mathbf{R}$. such that $\phi(u) = 1$ is called a state. A state ϕ is said to be extremal if, whenever φ is a positive homomorphism dominated by ϕ (i.e. $\varphi(g) \le \phi(g)$, for all g in G_+), then there is a non-negative real number λ such that $\lambda\phi = \varphi$.

Theorem [GH,G]. Let G be an ordered group with order unit. An element g of G is positive if $\phi(g)$ is strictly positive for all extremal states, ϕ, on G.

Theorem [E]. Let G be a dimension group which is the direct limit of a simple stationary system, i.e. a system of identical simplicial groups

$$H \xrightarrow{f} H \xrightarrow{f} H \xrightarrow{f} \dots \ ,$$

where f^r is strictly positive for some positive integer r. Then G has exactly one state.

For a unital C*-algebra A, the Grothendieck group $K_0(A)$ can be defined as follows [E]. Let $D_n(A)$ denote the set of equivalence classes of projections in the n x n matrix algebra over A, with respect to the von Neumann equivalence of projections. Let D(A) denote the direct limit of the system

$$D_1(A) \to D_2(A) \to D_3(A) \to \dots,$$

with the obvious inclusions. Then there is a natural binary operation on $D(A)$, with respect to which it is an abelian monoid. $K_0(A)$ is then the enveloping abelian group of $D(A)$. There is a natural map from $D(A)$ into $K_0(A)$, the image of which we denote by $K_0(A)_+$. $K_0(A)_+$ is not necessarily a positive cone in the ordered group sense, but a homomorphism $f: K_0(A) \to K_0(B)$, is said to be positive if it maps $K_0(A)_+$ into $K_0(B)_+$. We denote the class in $K_0(A)$ of the unit element 1_A, by $[1_A]$. Let $\phi: A \to B$ be a unital homomorphism of unital C*-algebras, then ϕ induces a unique positive homomorphism, $\phi_*: K_0(A) \to K_0(B)$, such that $\phi_*([1_A]) = [1_B]$.

If N is a multimatrix algebra $M(\mathbf{k})$, with $\mathbf{k} \in \mathbb{N}^n$, then $K_0(N)$ is isomorphic to $\mathbb{Z}^n$, and $K_0(N)_+$ is the simplicial ordering. If $M = M(\mathbf{l})$, with $\mathbf{l} \in \mathbb{N}^m$, and if $\phi: N \to M$ is a unital homomorphism, then $\phi_*: K_0(N) \to K_0(M)$ is given by the inclusion matrix A associated with ϕ.

Let $B = \overline{\cup B_n}$ be a unital AF algebra then [E]

$$K_0(B) = K_0(\cup B_n) \cong \varinjlim K_0(B_n),$$

and $K_0(B)_+$ corresponds, under this isomorphism, to the positive cone on the direct limit in the ordered group sense. Thus $K_0(B)$ is a dimension group. Since B is unital, the class of the unit, $[1_B]$, in $K_0(B)$, is an order unit.

Theorem [Ell].

(i) Given a countable dimension group G with order unit u, then there exists a unital AF C*-algebra B, and an order isomorphism $f: K_0(B) \to G$, such that $f([1_B]) = u$.

(ii) Let A, and B be unital AF algebras. Then A is isomorphic to B if, and only if there is an order isomorphism $f: K_0(A) \to K_0(B)$, such that $f([1_A]) = [1_B]$.

(iii) Let A, and B be unital AF algebras, and suppose there is a positive homomorphism $f: K_0(A) \to K_0(B)$, with $f([1_A]) = [1_B]$, then there exists a unital homomorphism $\phi: A \to B$, such that $\phi_* = f$. If $\operatorname{Ker} f \cap K_0(B)_+ = \{0\}$, then ϕ is injective.

(iv) If A, and B are unital AF algebras with $K_0(A) \cong K_0(B)$, then A, and B are stably isomorphic, i.e. $A \otimes K \cong B \otimes K$, where K denotes the compact operators on a separable Hilbert space.

Let Tr be an extremal, positive, normalized trace on a unital AF algebra A, then Tr_* is an extremal state on $K_0(A)$. Conversely to any extremal state p on $K_0(A)$, there is an extremal trace Tr on A with $Tr_* = p$.

The index of subfactors.

Let M be a type II_1 factor, and N a subfactor. Then M may be considered as a right N module. If it is finitely generated, then it is projective [PP], and thus defines a class in $K_0(N)$. The index of N in M, written $[M:N]$, is then the positive real number corresponding to the class of M in $K_0(N)$. Note that if M is of type II_1 then $K_0(N) \cong \mathbb{R}$. If M is not a finitely generated right N module, then the index of N in M is defined to be infinity. The index is a conjugacy invariant for N as a subfactor of M, and was introduced by V. Jones [J] in order to study subfactors of a type II_1 factor.

Suppose that M is a finite von Neumann algebra with a fixed normal, faithful, normalized trace, Tr. Let N be a von Neumann subalgebra of M with the same unit. Let E_N denote the unique conditional expectation of M onto N, defined by

$$Tr(E_N(x)y) = Tr(xy)$$

for $x \in M$, $y \in N$. Now if $L^2(M,Tr)$ denotes the Hilbert space of the GNS construction with respect to Tr, then M acts on $L^2(M,Tr)$ by left multiplication, and E_N extends naturally to an orthogonal projection, e_N on $L^2(M,Tr)$. The basic construction [J] for the pair $N \subseteq M$ is defined to be the von Neumann algebra acting on $L^2(M,Tr)$, generated by M and e_N, and is denoted by $\langle M,e_N\rangle$. Now $\langle M,e_N\rangle$ is a factor if, and only if N is a factor. Also $\langle M,e_N\rangle$ is finite if, and only if the index of N in M, $[M:N]$ is finite.

Now suppose that M is a finite factor, and N a subfactor with finite index, $[M:N]$ in M when M is of type II_1. Then since $\langle M,e_N\rangle$ is a factor, it has a unique normalized trace which we denote by Tr_1. The trace Tr_1, is an extension of Tr, and satisfies

$$Tr_1(xe_N) = [M:N]^{-1}Tr(x),$$

for x in M. Now the index of M in $\langle M,e_N\rangle$ is finite, and turns out to be the same as that for N in M. Thus one may iterate the basic construction to obtain a tower of finite factors

$$N \subseteq M = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \qquad (2.8),$$

where $M_1 = \langle M,e_N\rangle$, $M_2 = \langle M_1,e_M\rangle$, etc., with $[M_{i+1}:M_i] = [M:N]$ for $i = 0,1,2,\ldots$ .This construction also yields a sequence of orthogonal projections: $e_1 = e_N$, $e_2 = e_M$, e_3, ... , satisfying the relations:

$$e_n e_{n\pm1} e_n = \tau e_n \qquad (2.9),$$

$$e_n e_m = e_m e_n, \quad |m-n| > 1 \qquad (2.10),$$

$$Tr\,(xe_m) = \tau\, Tr\,(x), \quad x \in C^*(1,e_1,\ldots,e_{m-1}) \qquad (2.11),$$

where Tr denotes the obvious trace on the inductive limit of the tower of factors, and $\tau = [M:N]^{-1}$.

It was shown by V.Jones [J] that the C^*-algebra, $A(\tau)$, generated by a unit, and a sequence of projections, $\{e_n\}_{n=1}^{\infty}$ satisfying (2.9)-(2.11) for some real parameter τ, and trace Tr on $A(\tau)$, will be non-trivial only if

$$\tau^{-1} \in \{4\cos^2(\pi/\ell) : \ell = 3,4,\ldots\} \cup [4,\infty) \qquad (2.12).$$

Later Wenzl [Wen1] showed that the same was true, without assuming a trace on $A(\tau)$ satisfying (2.11). Jones [J] also showed that $A(\tau)$ is an AF algebra, and moreover that

$$A(\tau) \cong A(A_\ell) \quad \text{if } \tau^{-1} = 4\cos^2(\pi/\ell), \quad \ell = 3,4,\ldots \tag{2.13}$$

$$A(\tau) \cong A(A_\infty) \quad \text{if } \tau^{-1} \geq 4 \tag{2.14}$$

where A_l, $3 \leq l \leq \infty$, denote the usual Dynkin diagrams (see Figures 2,and 6). It follows that if N is a subfactor of a type II_1 factor M, then the index, $[M:N] = \tau^{-1}$, satisfies (2.12), or is infinite. Note also that the von Neumann algebra generated by a sequence of projections, $\{e_n\}_{n=1}^{\infty}$ satisfying (2.9)-(2.11) for τ as in (2.12), is a type II_1 factor, and the subfactor generated by $\{e_n\}_{n=2}^{\infty}$ has index τ^{-1}.

Given a subfactor of a type II_1 factor, with finite index, one can also construct a tunnel of subfactors. Specifically [J], if N is a subfactor of a type II_1 factor M, with finite index [M:N], then there is a subfactor N_1 of N, and a projection e_0 in M, such that M is isomorphic to the basic construction for the pair $N_1 \subseteq N$. It follows that the index for the pair $N_1 \subseteq N$, is the same as that for the pair $N \subseteq M$. The subfactor N_1 is unique up to conjugacy by unitary elements of N. The tunnel of subfactors [PP]

$$M \supseteq N = N_0 \supseteq N_1 \supseteq N_2 \supseteq \ldots$$

is obtained by iterating the above construction, and satisfies $[N_i:N_{i+1}] = [M:N]$, for $i = 0,1,2,\ldots$. Moreover there is a sequence of projections, $e_{-k} \in N_{k-1}$, for $k = 0,1,2,\ldots$, satisfying the relations (2.9)-(2.11).

We now give a formula for the index due to Pimsner, and Popa [PP]. Let M be a finite von Neumann algebra, with von Neumann subalgebras, $N_1 \subseteq N_2$, sharing the same unit as M. Let Tr be a faithful, normal, normalized trace on M, and let E_{N_1} be the trace preserving conditional expectation of N_2 onto N_1. Then put

$$\lambda[N_2:N_1] = \max \{ \lambda \geq 0 \,;\, E_{N_1}(x) \geq \lambda x,\ x \in N_{2+} \},$$

where N_{2+} denotes the positive cone of N_2. If M is a type II_1 factor, and N a subfactor, then $\lambda[M:N] = [M:N]^{-1}$. Suppose that $N = M(k)$, and $M = M(l)$ are multimatrix algebras with $k \in \mathbb{N}^n$, $l \in \mathbb{N}^m$, and there is an inclusion of N into

M given by the matrix $A = (a_{i,j})$, with $Ak = l$. Let the traces on M, and N be given by t, and s respectively, with $A^T t = s$, then

$$\lambda[M:N]^{-1} = \max \{ \sum_{j=1}^{n} (b_{i,j} s_j / t_i) \; ; \; i = 1,\dots,m \; \}$$

where $b_{i,j} = \min \{ a_{i,j}, k_j \}$.

When the index of a pair of II_1 factors lies in the discrete part of the range, i.e. when the index is less than four, the pair have the further property that

$$N' \cap M = \mathbb{C}1.$$

A subfactor with this property is said to be irreducible. It is an open problem to determine all values of the index greater than four, for which there exist irreducible subfactors. For all known examples of irreducible subfactors the index is an algebraic integer, and the smallest known value greater than four is $(5 + \sqrt{13})/2$ [HS].

Another major problem in the index theory for subfactors is that of determining all subfactors of the hyperfinite type II_1 factor with a given value of index. This leads to the introduction of finer invariants, which we now describe. Let N be a subfactor of a finite factor M, then starting with the tower (2.8), one can form the tower

$$\mathbb{C}1 = N' \cap N \subseteq N' \cap M_0 \subseteq N' \cap M_1 \subseteq N' \cap M_2 \subseteq \cdots,$$

which is called the derived tower [GHJ]. The von Neumann algebra $N' \cap M_k$, $k = 1,2,\dots$ is finite dimensional, with dimension bounded above by $[M:N]^k$. For $k = 1,2,\dots$, $N' \cap M_k$, also contains the the unit, and projections $e_1,\dots,e_k$. Thus the derived tower determines an AF algebra B, and by [GHJ,O], it is known that $B \cong A(\Gamma)$, where Γ is a connected, bipartite graph of norm $[M:N]^{1/2}$, with a distinguished vertex $*$. The graph Γ is a conjugacy invariant for the pair $N \subseteq M$, and is called the principal graph. The depth of the pair $N \subseteq M$, is defined to be the radius of the graph Γ from $*$. For a pair of type II_1 factors, with index less than four, the principal graph has to be a Dynkin diagram of type A-D-E, and so the pair will be of finite depth. If the pair is of finite depth, and the index is equal to four, then the principal graph has to be an extended Dynkin diagram of type A-D-E.

Consider the diagram

$$\begin{array}{ccc} P & \rightarrow & M \\ \uparrow & & \uparrow \\ Q & \rightarrow & N \end{array} \qquad (2.15),$$

where $M, N, P, Q,$ are finite von Neumann algebras, and the arrows denote unital inclusions. Suppose that there is a faithful, normal trace, Tr on M, and let E_N, E_P, and E_Q be the trace preserving conditional expectations onto N, P, and Q respectively. The diagram (2.15) is called a commuting square [PP,GHJ] if

$$E_P E_N = E_Q.$$

Let $B = \overline{\cup B_k}$ be a unital AF algebra, and let Tr be an extremal, faithful, positive trace on B. Let π denote the GNS representation of B with respect to Tr. Then the weak closure, M, of $\pi(B)$ is isomorphic to the hyperfinite type II_1 factor. Now let $C = \overline{\cup C_k}$ be a unital AF algebra, and suppose that the diagram

$$\begin{array}{ccc} C_{k+1} & \rightarrow & B_{k+1} \\ \uparrow & & \uparrow \\ C_k & \rightarrow & B_k \end{array} \qquad (2.16)$$

commutes, for all k, in the ordinary sense, where the arrows denote unital inclusions. Then it follows that there is a unital inclusion $j: C \rightarrow B$. If the restriction of trace to $j(C)$ is extremal, then the weak closure, N, of $\pi(j(C))$ will be a subfactor of M. The index of N in M, $[M:N]$, satisfies the inequality [PP]

$$[M:N]^{-1} \geq \operatorname{Lim}_k \sup \lambda[B_k : C_k].$$

Now suppose that the diagram (2.16) is a commuting square, i.e.

$$E_{C_{k+1}} E_{B_k} = E_{C_k}$$

for all k, where $E_{C_{k+1}}$, E_{B_k}, and E_{C_k} denote the trace preserving conditional expectations onto C_{k+1}, B_k, and C_k respectively, then

$$[M:N]^{-1} = \lim_k \lambda[B_k : C_k] .$$

One way of classifying subfactors of the hyperfinite type II_1 factor R, is by finite dimensional approximation, as above. The following Theorem states that this is possible in the finite depth case.

Theorem [O,P]. Let $N \subseteq M$, be a pair of hyperfinite type II_1 factors, with finite index, and finite depth. Then the tunnel of subfactors may be chosen so that

$$N = \overline{\cup N_k' \cap N}^{w},$$

$$M = \overline{\cup N_k' \cap M}^{w},$$

where w denotes weak closure. Moreover the diagram

$$\begin{array}{ccc} N_{k+1}' \cap N & \rightarrow & N_{k+1}' \cap M \\ \uparrow & & \uparrow \\ N_k' \cap N & \rightarrow & N_k' \cap M \end{array}$$

is a commuting square of finite dimensional C^*-algebras for all k, and there is a k_0, such that the diagram for k_0 completely determines all subsequent diagrams, and hence the pair $N \subseteq M$.

Thus the problem of classifying subfactors of finite index, and finite depth, reduces to that of classifying commuting squares of multimatrix algebras. This Theorem was originally announced by Ocneanu [O], with the additional assumption of trivial relative commutant, and used by him to classify subfactors of R with index less than four. Such subfactors will be irreducible, and of finite depth, and there is one with principal graph A_n, for $n = 3,4,\ldots$, and D_{2n}, for $n = 2,3,\ldots$, and there are two anticonjugate, but non-conjugate for the E_6, and E_8 graphs.

§3. The dimension group of $A(\Gamma)$.

Let Γ be an infinite connected graph with distinguished vertex $*$. For certain graphs Γ, we will define a family of polynomials $\{Q_v\}$, indexed by the vertices v of Γ. These give a family of Markov traces on $A(\Gamma)$, which will be used to determine the dimension group of $A(\Gamma)$.

If Δ is the incidence matrix of Γ, we will aim to find a family $\{\phi_v : v \in \Gamma^{(o)}\}$ of rational functions in an indeterminate x satisfying

$$x\phi_v = \sum_{w \in \Gamma^{(o)}} \Delta(v,w)\phi_w \qquad (3.1),$$

$$\phi_* = 1 \qquad (3.2).$$

Lemma 3.1.

Consider the graph $\Gamma = T_{p,q,\infty}$ with $p \geq q \geq 1$, and $* = 0$ (see Figure 1). Then functions $\{\phi_v\}$ satisfying (3.1) and (3.2) exist and are unique. They are

$$\phi_r = S_r, \quad 0 \leq r \leq p-1$$

$$\phi_p = S_{p+q-1}/S_{q-1}$$

$$\phi_{\bar{r}} = S_{p+q-2-r}\, S_{p-1}/S_{q-1}, \quad \text{if } q \geq 2, \ p \leq r \leq p+q-2$$

$$\phi_r = x\phi_{r-1} - \phi_{r-2}, \quad r \geq p+1, \qquad (3.3),$$

where $S_n \in \mathbb{Z}[x]$ are Chebyshev polynomials of the second kind satisfying

$$S_r = xS_{r-1} - S_{r-2}, \quad S_o = 1, \quad S_{-1} = 0 \qquad (3.4).$$

Proof:

We have $\phi_o = 1$, by (3.2), and for $v = 1$, we have by (3.1) $\phi_1 = x\phi_o = x$. Note that if $\{\chi_n\}_{n=0}^{N}$ is a set of rational functions satisfying

$$\chi_n = x\chi_{n-1} - \chi_{n-2} \qquad (3.5),$$

for $n = 2, \ldots, N$ and initial conditions, $\chi_0 = R$, $\chi_1 = xR$ then

$$\chi_n = RS_n, \quad n = 0, \ldots, N \tag{3.6}$$

Thus since $\phi_0 = 1$, $\phi_1 = x$, and also by (3.1)

$$\phi_i = x\phi_{i-1} - \phi_{i-2}$$

for $i = 2, \ldots, p-1$, we have

$$\phi_i = S_i, \quad i = 0, 1, 2, \ldots, p-1 \tag{3.7}$$

Similarly if $\phi_{\overline{p+q-2}} = R$, then $\phi_{\overline{p+q-3}} = x\phi_{\overline{p+q-2}} = xR$,

$$\chi_r = x\chi_{r-1} - \chi_{r-2}, \quad r = 2, \ldots, q-2 \tag{3.8}$$

Consequently from (3.6) we see that

$$\phi_{\overline{p+q-2-r}} = \chi_r = RS_r, \quad r = 0, \ldots, q-2 \tag{3.9}$$

To determine R we note from (3.1) that

$$x\phi_{\overline{p}} = \phi_{\overline{p+1}} + \phi_{p-1} \tag{3.10}$$

But $\phi_{\overline{p}} = RS_{q-2}$, $\phi_{\overline{p+1}} = RS_{q-3}$, $\phi_{p-1} = S_{p-1}$, thus

$$RS_{q-1} = R(xS_{q-2} - S_{q-3}) = S_{p-1}$$

and so

$$R = S_{p-1}/S_{q-1} \tag{3.11}$$

On the infinite arm of the graph we have

$$x\phi_{p-1} = \phi_{p-2} + \phi_{\overline{p}} + \phi_p \tag{3.12}$$

and so

$$\begin{aligned}\phi_p &= x\phi_{p-1} - \phi_{p-2} - \phi_{\overline{p}} = xS_{p-1} - S_{p-2} - S_{q-2}\,S_{p-1}/S_{q-1} \\ &= S_p - S_{q-2}S_{p-1}/S_{q-1} = S_{p+q-1}/S_{q-1}\end{aligned} \tag{3.13}$$

Here we have used the identity

$$S_{p+q-1} = S_pS_{q-1} - S_{q-2}S_{p-1} \tag{3.14}$$

which follows from interpreting S_i as the characteristic polynomial of A_i. The remainder is clear.

Let $\{\phi_v\}$ be a family of rational functions associated to a graph Γ, satisfying (3.1) and (3.2). Then we define, for $v \in \Gamma^{(o)}$:

$$Q_v(t) = x^{-d(v)}\phi_v(x) \qquad (3.15),$$

where $t = x^{-2}$. Then for $\Gamma = T_{p,q,\infty}$, $p \geq q \geq 2$

$$\begin{aligned}
&Q_r = P_r \quad 0 \leq r \leq p-1 \\
&Q_p = P_p - tP_{p-1}P_{q-2}/P_{q-1} = P_{p+q-1}/P_{q-1} \\
&Q_r = t^{r+1-p}\, P_{p-1}P_{p+q-2-r}\,/\,P_{q-1}, \text{ (if } q \geq 2) \ \ p \leq r \leq p+q-2 \\
&Q_r = Q_{r-1} - tQ_{r-2}, \quad r \geq p+1
\end{aligned} \qquad (3.16),$$

where $P_r \in \mathbb{Z}[t]$, $r = 0,1,2, \ldots\ldots$ are defined by

$$P_r(t) = x^{-r}S_r(x) \qquad (3.17),$$

with $t = x^{-2}$, and are the Jones' polynomials [J]:

$$P_r = P_{r-1} - tP_{r-2}, \quad P_o = 1, \ P_{-1} = 0 \qquad (3.18).$$

Lemma 3.2.

For $p,q \geq 3$, $P_{q-1} \mid P_{p-1}$ in $\mathbb{Z}[t]$ if, and only if $q \mid p$.

Proof:

The zeros of P_n are $\{(4\cos^2\left(\frac{k\pi}{n+1}\right))^{-1}\}_{k=1}^{[n/2]}$, and $P_n \mid P_m$ if, and only if the zeros of P_n are a subset of those for P_m. Thus if $P_{q-1} \mid P_{p-1}$, then $\frac{1}{q} = \frac{s}{p}$ for some integer s, $1 \leq s \leq [(p-1)/2]$, i.e. $p = sq$, and so $q \mid p$. Conversely, if $q \mid p$, then the set $\{k/q\}_{k=1}^{[\frac{q-1}{2}]}$ is clearly contained in the set $\{r/p\}_{r=1}^{[\frac{p-1}{2}]}$. Hence the zeros of P_{q-1} are also zeros of P_{p-1}, and so $P_{q-1} \mid P_{p-1}$.

Lemma 3.3.

$Q_v(t) \in \mathbb{Z}[t]$ for all $v \in T^{(0)}_{p,q,\infty}$, if and only if $q = 2$, or $q \mid p$.

Proof:

In the first place, if $q = 2$, then $P_{q-1} = 1$, and so clearly $Q_v \in \mathbb{Z}[t]$ for all v, since $P_k \in \mathbb{Z}[t]$ for all k. Now if $q \mid p$, then since $P_{q-1} \mid P_{p-1}$ we see that

$$Q_p = P_p - tP_{q-2}P_{p-1}/P_{q-1} \in \mathbb{Z}[t]$$

and

$$Q_r = t^{r+1-p}\, P_{p-1}P_{p+q-2-r} / P_{q-1} \in \mathbb{Z}[t]$$

for $r = p,\dots,p+q-2$. Then in view of the recurrence relation

$$Q_r = Q_{r-1} - t\,Q_{r-2}$$

for $r \geq p + 1$, and the fact that $Q_{p-1}, Q_p \in \mathbb{Z}[t]$, we have $Q_r \in \mathbb{Z}[t]$ for all $r \geq p - 1$.

Conversely, if $Q_v(t) \in \mathbb{Z}[t]$, for all v, we see by examining $Q_{\bar{r}}$, that $P_{q-1} \mid P_{p-1}P_{p+q-2-r}$ for $r = p,\dots,p+q-2$, but if $q > 2$, P_{q-1} does not divide P_{q-3} since $\deg P_{q-3} < \deg P_{q-1}$. Then since P_{q-3} and P_{q-1} have no common zeros, $P_{q-1} \mid P_{p-1}$ and so by the above lemma $q \mid p$.

Lemma 3.4.

Let $\{Q_v\}$ be the rational functions associated with the graph $T_{p,q,\infty}$ for $p \geq q$. Let γ_r, γ'_r denote the smallest zero and second smallest zero respectively of Q_r, for $r \geq p$.

(a) $Q_r(0) = 1$ for $r \geq p$.

(b) The sequence $(\gamma_r)_{r\geq p}$ is strictly decreasing and converges to $\gamma_{p,q} = ||T_{p,q,\infty}||^{-2} > 0$.

(c) The zeros of Q_r lie in the interval $[\gamma_r, \infty)$ for $r \geq p$, the poles (if any) lie in the interval (γ'_r, ∞).

(d) $\gamma_{r+1} < \gamma_r < \gamma'_{r+1}$ for $r \geq p$.

Proof:

Let $\bar{\Delta}_r = S_{q-1}\phi_r$, for $r \geq p$. Then $\bar{\Delta}_p = S_{p+q-1}$ is the characteristic polynomial of the graph A_{p+q-1}, and we claim that for $r > p$, $\bar{\Delta}_r$ is the characteristic polynomial Δ_r of the graph $T_{p,q,r-p+1}$. In the first place, $S_{q-1}\phi_{p+1}$, $S_{q-1}\phi_{p+2}$ are the characteristic polynomials of the graphs $T_{p,q,2}$ and $T_{p,q,3}$ respectively, by inspection. Moreover by expansion of the determinant defining Δ_r we have

$$\Delta_r = x\Delta_{r-1} - \Delta_{r-2}, \text{ for } r \geq p+3 \qquad (3.19).$$

But $\bar{\Delta}_r$ satisfies the same recurrence relation (3.19) for $r \geq p+3$ as does Δ_r. Hence since $\bar{\Delta}_{p+1} = \Delta_{p+1}$ and $\bar{\Delta}_{p+2} = \Delta_{p+2}$ we have $\Delta_r = \bar{\Delta}_r$ for $r \geq p+1$. Thus the polynomials $\bar{\Delta}_r$ for $r \geq p$ are the characteristic polynomials of an ascending sequence of subgraphs of $T_{p,q,\infty}$:

$$A_{p+q-1} \subseteq T_{p,q,2} \subseteq T_{p,q,3} \subseteq \ \ldots\ldots$$

Since the incidence matrices are self-adjoint, all the roots of Δ_r are real, and if β_r is the largest root, it is the Perron-Frobenius eigenvalue of the corresponding incidence matrix, and all the roots of Δ_r lie in $[-\beta_r, \beta_r]$. Moreover if β'_r is the second largest root, then we have the interlacing property [CDS, Theorem 0.10, p19]:

$$\beta'_r \le \beta'_{r+1} \le \beta_r \le \beta_{r+1}, \quad \text{for } r \ge p \qquad (3.20).$$

The above sequence of graphs is strictly increasing and so the sequence $\{\beta_r\}_{r\ge p}$ is strictly increasing and converges to $\lambda_{p,q} = ||T_{p,q,\infty}||$, where $2 \le \lambda_{p,q} \le 3$. Note also that the sequence $\{\beta'_r\}$ is also increasing by interlacing.

We now show that $\beta_r > \beta'_{r+1}$ for $r \ge p$. Thus, suppose $\beta_n = \beta'_{n+1}$ for some $n \ge p+1$. Then we have

$$0 = \bar{\Delta}_{n+1}(\beta'_{n+1}) = \beta'_{n+1}\,\bar{\Delta}_n(\beta'_{n+1}) - \bar{\Delta}_{n-1}(\beta'_{n+1})$$
$$= \beta'_{n+1}\,\bar{\Delta}_n(\beta_n) - \bar{\Delta}_{n-1}(\beta_n) = -\bar{\Delta}_{n-1}(\beta_n).$$

But $\beta_n > \beta_{n-1}$, where β_{n-1} is the largest root of $\bar{\Delta}_{n-1}$, giving a contradiction. It remains to show that $\beta_p > \beta'_{p+1}$. For this, note that

$$\bar{\Delta}_{p+1} = x\bar{\Delta}_p - \bar{\Delta}_{p-1}$$

where $\bar{\Delta}_{p-1} = S_{q-1}S_{p-1}$, which has $2\cos(\frac{\pi}{p})$ as its largest root. Then if $\beta'_{p+1} = 2\cos(\frac{\pi}{p+q}) = \beta_p$ we again obtain a contradiction. Hence we have $\beta'_{r+1} < \beta_r < \beta_{r+1}$ for all $r \ge p$.

Now when $p = q$, $\phi_p = S_{2p-1}/S_{p-1} \in \mathbb{Z}[x]$, with largest root $\beta_p = 2\cos\frac{\pi}{2p}$, and second largest root $2\cos(\frac{3\pi}{2p})$, less than β'_p. In addition $\phi_r \in \mathbb{Z}[x]$, for all $r \ge p$, with largest root β_r, and second largest root less than or equal to β'_r, and all zeros lie in the interval $[-\beta_r, \beta_r]$. Suppose that $p > q$. For $r \ge p$, $\phi_r = \Delta_r/S_{q-1}$, and so the poles of ϕ_r will be a subset of the roots of S_{q-1}. Now $2\cos(\frac{\pi}{q})$ is the largest zero of S_{q-1}, and all its zeros are simple and lie in the interval $I = [-2\cos(\pi/q), 2\cos(\pi/q)]$. Now, $2/(p+q) < 1/q$, and so $\beta'_p = 2\cos(\frac{2\pi}{p+q}) > 2\cos(\frac{\pi}{q})$. Hence $\beta'_r \ge \beta'_p > 2\cos(\frac{\pi}{q})$, for all $r \ge p$. Hence $I \subseteq (-\beta'_r, \beta'_r)$, and the largest, and second largest roots of ϕ_r are β_r, and β'_r respectively, for $r \ge p$.

By inspection $Q_p(0) = 1 = Q_{p+1}(0)$, and since $Q_r = Q_{r-1} - t\,Q_{r-2}$, for $r \ge p+2$, we see that $Q_r(0) = 1$ for all $r \ge p$. Thus if a is a zero of Q_r for $r \ge p$, then $a \ne 0$, and from $Q_r(t) = x^{-r}\phi_r(x)$, $t = x^{-2}$, we see that $b = a^{-\frac{1}{2}}$ is a zero of ϕ_r. Conversely, if $b \neq 0$ is a zero of ϕ_r, then b^{-2} is a zero of Q_r. Then if $p > q$ since $\beta_r > \beta_r' > 2\cos(\pi/q)$ for $r \geq p$, we see that the smallest and second smallest roots of Q_r are $\gamma_r = \beta_r^{-2}$, and $\gamma_r' = \beta_r'^{-2}$ respectively, that the zeros of Q_r are real and lie in the interval $[\gamma_r,\infty)$ and that the poles lie in the interval $[(2\cos\pi/q)^{-2},\infty)$ where $\gamma_r < (2\cos\frac{\pi}{q})^{-2}$. When $p = q$, Q_r has no poles, and the smallest zero of Q_r is $\gamma_r = \beta_r^{-2}$, and the second smallest is $\gamma_r' \geq \beta_r'^{-2}$. Thus the lemma follows from the preceding with $\gamma_{p,q} = \lambda_{p,q}^{-2}$, and $\gamma_r = \beta_r^{-2}$, $r \geq p$.

Lemma 3.5.

Let $\{Q_v\}$ be the rational functions associated with the graph $T_{p,q,\infty}$, $p \geq q \geq 1$. Then

$$\{t \geq 0 : Q_v(t) \geq 0, \forall v \in T^{(0)}_{p,q,\infty}\} = [0,\gamma_{p,q}], \tag{3.21}$$

Proof:

For $q = 1$, $T_{p,1,\infty} = A_\infty$, and so recall from [J] that

$$\{t \geq 0 : P_n(t) > 0,\ n = 0,1,2,\ldots\} = [\,0,\tfrac{1}{4}\,] \tag{3.22}$$

Note also that $\frac{1}{9} \leq \gamma_{p,q} \leq \frac{1}{4}$. Then since $Q_i = P_i$ for $0 \leq i \leq p-1$ we see that $Q_i(t) > 0$ for $t \in [0,\gamma_{p,q}]$ and $0 \leq i \leq p-1$. Also for $p \leq r \leq p+q-2$, we have

$$Q_{\bar{r}} = t^{r+1-p}\, P_{p-1}P_{p+q-2-r}/P_{q-1} \tag{3.23}$$

which is strictly positive on the interval $(0,\gamma_{p,q}]$ and $Q_{\bar{r}}(0) = 0$. Moreover, by Lemma 3.4, the zeros and poles of $\{Q_r : r \geq p\}$ lie in the interval $(\gamma_{p,q},\infty)$, and then since $Q_r(0) = 1$, these rational functions are strictly positive on the interval $[0,\gamma_{p,q}]$. Hence the right hand side of (3.21) is contained in the left hand side.

For the reverse inclusion, we take $\mu > \gamma_{p,q}$, and we show that there exists some $v \in T^{(0)}_{p,q,\infty}$, such that $Q_v(\mu) < 0$. There are four cases to consider:

(i) $\gamma_{p,q} < \mu \leq \gamma_p$.

By lemma 3.4, $\gamma_n \downarrow \gamma_{p,q}$ and so we can choose, $n \geq p + 1$, such that $\gamma_n < \mu \leq \gamma_{n-1}$. Then by the interlacing property 3.4(d), we have $\gamma_n < \gamma_{n-1} < \gamma_n'$ But γ_n is the smallest root of Q_n and $Q_n(0) = 1$. Thus Q_n is positive on $[0, \gamma_n)$ and so $Q_n(\mu) < 0$.

(ii) $\gamma_p < \mu < \gamma_p'$.

Again since $Q_p(0) = 1$, we have $Q_p(\mu) < 0$.

(iii) $\gamma_p' \leq \mu < 1$.

Suppose that $p > q$, then we have $(q + p)/2 < p$. Thus

$$\gamma_p' = \beta_p'^{-2} = (2 \cos \frac{2\pi}{p + q})^{-2} > (2 \cos \pi/p)^{-2}.$$

If $p = q$, then $\gamma_p' = (2 \cos \frac{3\pi}{2p})^{-2} > (2 \cos \frac{\pi}{p})^{-2}$. But $(2 \cos (\pi/p))^{-2}$ is the smallest root of Q_{p-1}, and Q_i is the Jones polynomial P_i for $i = 0,1,\ldots,p-1$, and so we can use an argument similar to (a) to show that there is an r, $0 \leq r \leq p - 1$ such that $Q_r(\mu) < 0$.

(iv) $\mu \geq 1$.

If $p = q = 2$, then $Q_2(t) = 1 - 2t$, so $Q_2(\mu) < 0$.

If $p = 3$, $q = 2$, then $Q_3(t) = P_4(t) = 1 - 3t + t^2$, so $Q_3(1) < 0$; also $Q_2(t) = 1 - t$, so $Q_2(\mu) < 0$ for $\mu > 1$.

If $p = 3$. $q = 3$. then $Q_3 = P_5/P_2 = 1 - 3t$, so $Q_3(\mu) < 0$.

If $p > 3$, then $Q_3(t) = 1 - 2t$, so $Q_3(\mu) < 0$.

The vertices of the graph $\hat{T}_{p,2,\infty}$, are labelled as in figure 3.

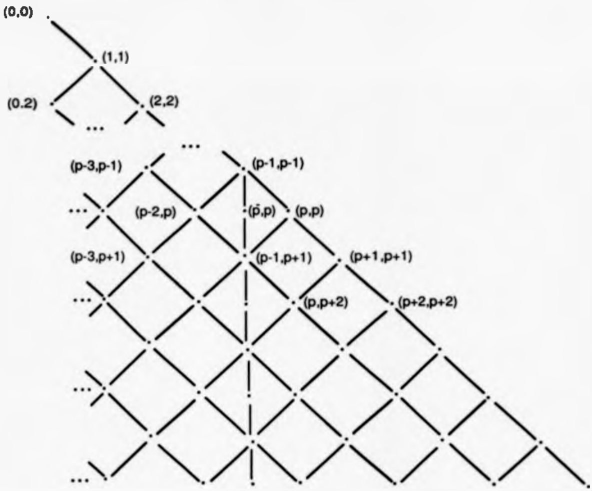

Figure 3 : $\hat{T}_{p,2,\infty}$

We associate to each vertex (v,n) of $\hat{T}_{p,q,\infty}$, the polynomial

$$Q_{(v,n)}(t) = t^{(n-d(v))/2} Q_v(t) \tag{3.24}$$

where d is the distance function on $\hat{T}_{p,q,\infty}$. Thus our notation is consistent with the embedding of $T_{p,q,\infty}$ in $\hat{T}_{p,q,\infty}$. Also for a connected graph Γ, and $v \in \Gamma^{(0)}$, we let $P^k(v)$ denote the set of paths of length k from v. For $i \in P^k(v)$, let $r(i)$ denote the endpoint. Note that since Γ is connected there is at least one $i \in P^2(v)$ such that $r(i) = v$.

Lemma 3.6.

Consider the graph $T_{p,q,\infty}$, for $p \geq q = 2$, with associated polynomials $\{Q_{(v,n)} : (v,n) \in \hat{T}^{(0)}_{(p,2,\infty)}\}$.

(a) The polynomials $\{Q_{(v,n)}\}$ satisfy the following splitting rules:

$$Q_{(v,n)} = \sum_{w} \Delta(v,w) Q_{(w,n+1)}.$$

(b) The linear span over $\mathbb{Z}$ of the set of polynomials $\{Q_{(v,n)} : (v,n) \in T^{(0)}_{p,2,\infty}\}$ is $\mathbb{Z}[t]$.

(c) If p is even, then for each fixed k, the set of polynomials associated to level $2k+1$ of $\hat{T}_{p,2,\infty}$ are linearly independent over $\mathbb{Z}$. If p is odd, those associated with level $2k$ are linearly independent over $\mathbb{Z}$.

(d) $$(1-t)\, Q_{(v,n)} = \sum_{i \in P^2(v)}{}' \, Q_{(r(i),n+2)}$$

where Σ' means that we omit the vertex v exactly once from the summation.

Proof:

(a) is clear.

(b) and (c): Suppose p is even, then there are $k+1$ vertices on level $2k+1$ of $\hat{T}_{p,2,\infty}$. We prove that if the monomials $\{t^j\}_{j=0}^{k}$ are in the linear span over $\mathbb{Z}$ of the polynomials associated with level $2k+1$ of $\hat{T}_{p,2,\infty}$ (for level 1 this is clear), then $\{t^j\}_{j=0}^{k+1}$ are in the corresponding linear span for level $2k+3$ of $\hat{T}_{p,2,\infty}$. This is enough to establish (b) and (c) by induction. Suppose that for level $2k+1$ we have

$$t^j = \sum_{\ell} a_{j\ell}\, Q_{(2\ell+1,2k+1)} \qquad j = 0,1,2,\ldots,k \tag{3.25}$$

where $a_{j\ell} \in \mathbb{Z}$. Then using the splitting rules, we express each polynomial $Q_{(2\ell+1,2k+1)}$ in (3.25) as a sum of the polynomials associated with level

$2k+2$, then we repeat to get each of those on level $2k+2$, as a sum of those on level $2k+3$. We also have

$$t^{k+1} = Q_{(1,2k+3)}$$

and so the set $\{t_j\}_{j=0}^{k+1}$ is in the desired linear span. The case when p is odd is similar.

(d) If $(u,v) \in T^{(0)}_{p,q,\infty}$, we write $u \sim v$ if $(u,v) \in T^{(1)}_{p,2,\infty}$. Then from the definitions of the polynomials ϕ_v we have

$$\phi_2\phi_v = (x^2-1)\phi_v = x\Big(\sum_{u\sim v}\phi_u\Big) - \phi_v = \sum_{u\sim v}\sum_{w\sim u}\phi_w - \phi_v = \sum_{w\in P^2(v)}{}' \phi_w$$

Now $Q_v(t) = x^{-d(v)}\phi_v(x)$, where $t = x^{-2}$, and so

$$\begin{aligned} Q_2Q_{(v,n)} &= t^{(n-d(v))/2}Q_2Q_v = x^{-n-2}\phi_2\phi_v = x^{-n-2}\phi_w = \sum_{w\in P^2(v)}{}'\phi_w \\ &= x^{-n-2}\sum_w x^{d(w)}Q_w(t) = \sum_{w\in P^2(v)}{}' Q_{(w,n+2)}(t). \end{aligned}$$

Let $F_n = A(T_{p,2,\infty})_n$ denote the finite dimensional algebra corresponding to level n of the Bratteli diagram of $A(T_{p,2,\infty})$ and $j_n : F_n \to F_{n+1}$ the corresponding inclusion. Thus if p is even $K_0(F_{2k+1})$ is isomorphic to $\mathbb{Z}^{k+1}$, and we can define maps $\theta_{2k+1} : K_o(F_{2k+1}) \to \mathbb{Z}[t]$ by

$$\theta_{2k+1}[(a_i)_{i=0}^{k}] = \sum_{j=0}^{k} a_j\, Q_{(2j+1,2k+1)} \tag{3.26}$$

which are injective by Lemma 3.6(c). It follows from Lemma 3.6(a) that the diagram

$$\begin{tikzcd} K_o(F_{2k+3}) \arrow[dr, "\theta_{2k+3}"] & \\ & \mathbb{Z}[t] \\ K_o(F_{2k+1}) \arrow[uu, "\iota_{2k+1}"] \arrow[ur, "\theta_{2k+1}"'] & \end{tikzcd} \tag{3.27}$$

is commutative, where $\zeta_n = K_o(j_{2k+2} \circ j_{2k+1})$. Hence using Lemma 3.6(b) we have an isomorphism

$$\theta_\infty : \lim_{\rightarrow} K_o(F_{2k+1}) \cong K_o(A(T_{p,2,\infty})) \rightarrow \mathbb{Z}[t] \qquad (3.28).$$

The case, p odd is similar, we have an isomorphism:

$$\theta_\infty : \lim_{\rightarrow} K_o(F_{2k}) \cong K_o(A(T_{p,2,\infty})) \rightarrow \mathbb{Z}[t] \qquad (3.29).$$

We now characterize the polynomials in $\mathbb{Z}[t]$ corresponding to the positive cone of $K_o(A(T_{p,2,\infty}))$. We let

$$R = \lim_{\mathbb{Z}} \{Q_{(r,n)}\} = \mathbb{Z}[t] \cong K_o(A(T_{p,2,\infty})) \qquad (3.30)$$

$$R_+ = \lim_{\mathbb{N}} \{Q_{(r,n)}\} \cong K_o(A(T_{p,2,\infty}))_+ \qquad (3.31)$$

Then R is a ring with an additively closed subset R_+ satisfying

$$R_+ - R_+ = R, \qquad R_+ \cap (-R_+) = \{0\} \qquad (3.32),$$

so that considered as a group, (R,R_+) is partially ordered. Since (R,R_+) is a dimension group it is unperforated. Note also that $1 = Q_{(0,0)} \in R_+$, and using Lemma 3.6(a), we can express $Q_{(0,0)}$ as a linear combination of polynomials associated to level n of $\hat{T}_{p,2,\infty}$, in which each polynomial occurs with a positive integer coefficient, for all $n \geq 0$. Thus $1 = Q_{(0,0)}$ is an order unit in (R,R_+), i.e. for all $P \in R_+$, there exists $k \in \mathbb{N}$ such that $k.1 - P \in R_+$. Later we will show that $R_+R_+ \subseteq R_+$, so that (R,R_+) is in fact a partially ordered commutative ring. Let $Q \in R_+$, then $Q = \sum a_{(r,n)} Q_{(r,n)}$, with $a_{(r,n)} \in \mathbb{N}$. Thus either $Q = 0$ or $Q(t) > 0$ for $t \in (0,\gamma_{p,2}]$. We will show that this property characterizes R_+, and consequently that $K_o(A(T_{p,2,\infty}))$ is an ordered ring.

Lemma 3.7.

Let C be a subset of $\mathbb{Z}[t]$, with $1 \in C$ such that $(\mathbb{Z}[t],C)$ is an ordered group and $tC \subseteq C$, $(1-t)C \subseteq C$. Then every normalized extremal state on $(\mathbb{Z}[t],C)$ is a ring homomorphism.

Proof:

Let ϕ be a normalized (i.e $\phi(1) = 1$) extremal state. Put $\varphi(P) = \phi(tP)$, for $P \in \mathbb{Z}[t]$. Then since $tC \subseteq C$, φ is a state on $(\mathbb{Z}[t],C)$. Suppose $P \in C$, then $(1-t)P \geq 0$, i.e. $P \geq tP$ and so $\phi(P) \geq \phi(tP) = \varphi(P)$. Hence φ is a positive homomorphism dominated by ϕ, but ϕ is extremal and so there exists a real non-negative number λ such that $\lambda\phi = \varphi$. Thus since ϕ is normalized we have

$$\lambda = \lambda\phi(1) = \varphi(1) = \phi(t.1) = \phi(t) \qquad (3.33).$$

It follows inductively from (3.33) that $\phi(t^n) = \lambda^n$ for all $n \geq 0$, hence $\phi(P) = P(\lambda)$ for all $P \in \mathbb{Z}[t]$.

Corollary 3.8.

If ϕ is a normalized extremal state on $K_0(A(T_{p,2,\infty}))$, then $\phi(P) = P(\lambda)$ for some $\lambda \in [0,\gamma_{p,2}]$, and all $P \in K_0(A(T_{p,2,\infty})) = \mathbb{Z}[t]$.

Proof:

It is clear from the definition of R_+ that $tR_+ \subseteq R_+$; moreover $(1-t)R_+ \subseteq R_+$ by Lemma 3.6(d). Since $t \in R_+$, we have $\phi(t) = \lambda \geq 0$. Moreover $Q_v \in R_+$ for all $v \in T^{(0)}_{p,2,\infty}$, and so $\phi(Q_v) = Q_v(\lambda) \geq 0$. Hence $\lambda \in [0,\gamma_{p,2}]$ by Lemma 3.5.

Theorem 3.9.

$K_o(A(T_{p,2,\infty})) \cong \mathbb{Z}[t]$, and $K_o(A(T_{p,2,\infty}))_+$ corresponds under this isomorphism to the set

$$\{0\} \cup \{Q \in \mathbb{Z}[t] : Q(\lambda) > 0 \text{ for } \lambda \in (0,\gamma_{p,2}]\}.$$

Proof:

It remains to show that if $Q \in \mathbb{Z}[t]$ and $Q(\lambda) > 0$ for $\lambda \in (0,\gamma_{p,2}]$, then Q is a linear combination of the polynomials $\{Q_{(v,n)} : (v,n) \in \hat{T}^{(0)}_{p,2,\infty}\}$ with non-negative integer coefficients. We can write $Q(t) = t^k P(t)$, for some non-negative integer k, where $P(\lambda) > 0$ for all $\lambda \in [0,\gamma_{p,2}]$. Let ϕ be any normalized extremal state on $K_o(A(T_{p,2,\infty}))$. Then by Corollary 3.8, $\phi(P) = P(\lambda)$ for some $\lambda \in [0,\gamma_{p,2}]$, and so $\phi(P) > 0$. It follows from [GH,G] that P is in $K_o(A(T_{p,2,\infty}))_+$ and hence so is $Q = t^k P$.

Corollary 3.10.

$K_o(A(T_{p,2,\infty}))$ is a partially ordered ring, for $p \geq 2$. In particular, Corollary 3.10 says that there exist non-negative integers $a^{\gamma}_{\alpha\beta}$ for $\alpha,\beta,\gamma \in \hat{T}^{(0)}_{p,2,\infty}$ such that

$$Q_\alpha Q_\beta = \sum_\gamma a^{\gamma}_{\alpha\beta} Q_\gamma$$

A special case of this has already been verified in Lemma 3.6(d).

§4. Embeddings.

Let Γ be a graph with incidence matrix $\Delta = [\Delta_{vw}]$ where the number of edges adjacent to a vertex is finite. We will consider the problem of concretely describing embeddings of $A(A_n)$ in $A(\Gamma)$ for some $2 \le n \le \infty$. From §3, we see that this is related to the question of expressing the generators of $K_o(A(A_\infty))$ namely the Chebyshev polynomials of the second kind S_n (associated to the graph A_∞) as linear combinations of the rational functions $\{\phi_v : v \in \Gamma^{(0)}\}$ associated to the graph Γ, with non-negative integer coefficients.

Let $V(\Gamma)$ denote the free module over $\mathbb{Z}$, generated by the vertices of Γ identifying an element $a \in V(\Gamma)$ as $a = (a_v)$, $a_v \in \mathbb{Z}$, $v \in \Gamma^{(0)}$. We let $V(\Gamma)_+$ be the positive cone generated by the vertices. If Γ is locally finite the incidence matrix Δ acts on $V(\Gamma)$. We seek a sequence $a_n = (a_{nv})$ in $V(\Gamma)_+$ with

$$S_n = \sum_{v \in \Gamma^{(0)}} a_{nv} \phi_v \qquad n = 0,1,2,\ldots \tag{4.1}$$

We claim that the a_n's can be found from the recurrence relations for (S_n) and (ϕ_v) as follows. In the first place $S_o = \phi_* = 1$, and so we can take $a_{o*} = 1$. Next we use the recurrence relations

$$xS_n = \Delta_{A_\infty} S_n, \qquad x\phi_v = \Delta_\Gamma \phi_v \tag{4.2}$$

to see that if $a_o, \ldots, a_n$ are given, then

$$xS_n = \sum_v a_{nv}\, x\phi_v = \sum_v a_{nv} \left(\sum_w \Delta_{vw} \phi_w\right) = \sum_w \left(\sum_v \Delta_{vw} a_{nv}\right) \phi_w \tag{4.3}$$

and so

$$S_{n+1} = xS_n - S_{n-1} = \sum_w \left(\sum_v \Delta_{vw}\, a_{nv} - a_{n-1w}\right)\phi_w \tag{4.4}$$

Thus we can take

$$a_{n+1} = \Delta a_n - a_{n-1} \quad \text{for } n \ge 0 \tag{4.5}$$

with $a_{-1} = 0$. The calculation (4.5) is illustrated in figure 4 for $\Gamma = E_\infty$.

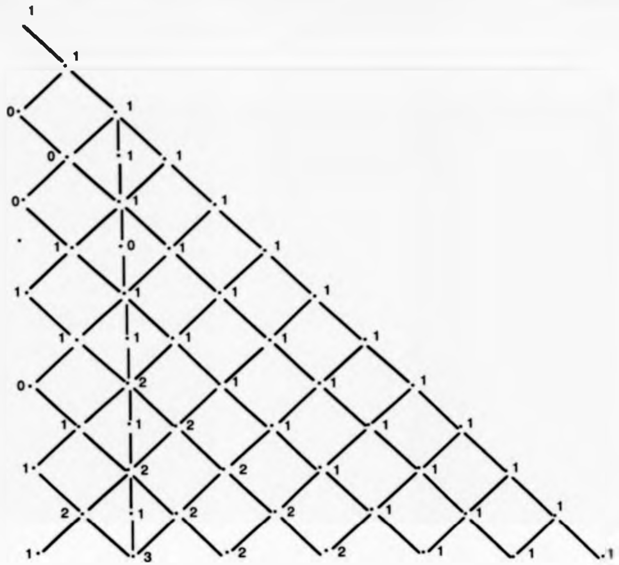

Figure 4

The numbers appearing in level n of $\hat{E}_\infty$ give the coefficients a_{nv} of the polynomial ϕ_v (corresponding to vertex v in $E_\infty^{(0)}$) in the expansion of the Chebyshev polynomial S_n. Note that if n is even (respectively odd) then only ϕ_v with $d(v)$ even (respectively odd) and $d(v) \leq n$ occurs in the expansion of S_n. This can be seen inductively to hold from (4.5) for any connected graph where there are no cycles of odd length. One obtains the coefficients at level $n+1$ by pushing forward on $\hat{E}_{\infty}$ those on level n to level $n+1$, then adding the values obtained at each vertex of level $n+1$, and then subtracting the number from the same vertex on level $n-1$. The question then arises of whether a_n, as defined above by (4.5) lie in $V(\Gamma)_+$ (i.e. whether $a_{nv} \geq 0$ for all n,v). If a_n in $V(\Gamma)$ satisfy $a_{-1} = 0$, $a_1 = 1$, and $a_{n+1} = \Delta a_n - a_{n-1}$, then a_n satisfy the same recurrence relation as the Chebyshev polynomials S_n and so $a_n = S_n(\Delta)\, a_o$ for $n \geq 0$. It is thus important to know when $S_n(\Delta) \geq 0$. Here a linear map $T = [T_{vw}]$, between $V(\Gamma_1)$ and $V(\Gamma_2)$ is said to be positive, written $T \geq 0$ if T maps $V(\Gamma_1)_+$ into $V(\Gamma_2)_+$, or equivalently $T_{vw} \geq 0$, for all v,w.

Lemma 4.1.

Let Γ be a finite bipartite connected graph. Then:

(a) If $||\Delta|| < 2$, then $S_m(\Delta) \gneq 0$ for $m = 0,1,2,\dots,\ell-1$, and $S_\ell(\Delta) = 0$ where $\ell \geq 2$ is given by $||\Delta|| = 2\cos(\pi/(\ell+1))$.

(b) If $||\Delta|| \geq 2$, then $S_m(\Delta) \gneq 0$ for all $m = 0,1,2,\dots$.

Proof:

We write $\Delta = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}$, partitioning the vertices according to their parity so that $\Delta_{ij} = 0$ unless i even, j odd, and vice-versa, and A^T denotes the transpose of A. Let $x_o \in V(\Gamma)$ and define $x_n = S_n(\Delta)x_o$, and $\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^n x_n$. Then

$$
\begin{aligned}
v_1 &= Av_o \\
v_2 &= A^T v_1 - v_o = (A^T A - 1)v_o \\
v_3 &= Av_2 - v_1 = A(A^T A - 1)v_o \\
v_n &= A_n v_{n-1} - v_{n-2} \qquad n \geq 1, \quad v_{-1} = 0
\end{aligned}
\tag{4.6}
$$

where $A_n = A$ if n is odd, and A^T if n is even. Similarly

$$u_n = A_n^T u_{n-1} - u_{n-2}, \quad n \geq 1, \quad u_{-1} = 0 \tag{4.7}$$

We recall from Hamachi [H], as AA^T is irreducible, that if λ denotes the largest eigenvalue of AA^T then:

(a) If λ is strictly less than 4, then $\lambda = 4\cos^2 \pi/(\ell + 1)$ for some $\ell \geq 2$, and if $v_0 \gneqq 0$, then $v_1 \gneqq 0, \ldots, v_{\ell-1} \gneqq 0$ and $v_\ell = 0$.

(b) If $\lambda \geq 4$, and $v_0 \neq 0$, then $v_m \gneqq 0$ for all m.

Since the maximum eigenvalue of A^TA is also λ, the sequence (u_n) will have the same behaviour as (v_n). Thus the lemma follows, by taking x_0 to be an arbitrary vertex of Γ.

Lemma 4.2.

Let Γ be a locally finite infinite connected bipartite graph. Then $S_m(\Delta) \gneqq 0$ for all $m = 0,1,2,\ldots$.

Proof:

Fix an arbitrary vertex u of Γ, and let $d_u(w)$ denote the distance of any vertex w in Γ from u. Let Γ_n denote the subgraph of Γ consisting of all vertices of distance less than or equal to n from u, and all edges in Γ joining these vertices. Let Δ_n be the incidence matrix of Γ_n, and extend Δ_n to an operator $\tilde{\Delta}_n$ on $V(\Gamma)$ by setting $\tilde{\Delta}_n v = 0$ if $v \notin \Gamma_n^{(0)}$, $\tilde{\Delta}_n v = \Delta v$ if $v \in \Gamma_n^{(0)}$. We claim that

$$S_m(\Delta)u = S_m(\tilde{\Delta}_n)u \quad \text{for} \quad m = 0,1,2,\ldots,n.$$

This is clear for $m = 0,1$. Now S_m is a polynomial of degree m. Thus if $0 \leq m < n$, we have

$$S_m(\tilde{\Delta}_n)u \in \operatorname{span}\{v \in \Gamma^{(0)} : d_u(v) \leq m\}.$$

Since $\Delta v = \tilde{\Delta}_n v$ if $d_u(v) < n$ we then have

$$\Delta S_m(\tilde{\Delta}_n)u = \tilde{\Delta}_n S_m(\tilde{\Delta}_n)u \quad \text{for} \quad 0 \leq m < n.$$

Then for $0 \leq m < n$, we see inductively that if $S_i(\Delta)u = S_i(\tilde{\Delta}_n)u$ holds for $i = 0,1,2,\ldots,m$ then

$$S_{m+1}(\tilde{\Delta}_n)u = \tilde{\Delta}_n S_m(\tilde{\Delta}_n)u - S_{m-1}(\tilde{\Delta}_n)u = \Delta S_m(\tilde{\Delta}_n)u - S_{m-1}(\tilde{\Delta}_n)u$$
$$= \Delta S_m(\Delta)u - S_{m-1}(\Delta)u = S_{m+1}(\Delta)u.$$

We apply the previous lemma to the subgraph Γ_n. If $\|\Gamma_n\| < 2$, then $\|\Gamma_n\| = 2\cos \pi/m$ for some m. But Γ_n has A_{n+1} as a subgraph, and from $\|A_{n+i}\| \leq \|\Gamma_n\|$ we deduce that $m \geq n+2$. Then by Lemma 4.1(a) we deduce that $S_r(\Delta_n)u \gneq 0$ for $r = 0,\ldots,n$ and in particular $S_n(\Delta_n)u \gneq 0$. We have already observed that $S_n(\tilde{\Delta}_n)u = S_n(\Delta)u$, and as $S_n(\tilde{\Delta}_n)u = \tilde{S}_n(\Delta_n)u$, we have $S_n(\Delta)u \gneq 0$. It only remains to consider the case $\|\Gamma_n\| \geq 2$, in which case $S_r(\Delta_n)u \gneq 0$ for all r and so $S_n(\Delta)u = S_n(\tilde{\Delta}_n)u \gneq 0$.

Lemma 4.3.

Let Γ_1 and Γ_2 be locally finite, infinite, bipartite graphs, with distinguished vertices $*_1$ and $*_2$, distance functions d_1 and d_2, and incidence matrices Δ_1 and Δ_2 respectively. Suppose that $\{\varphi_v : v \in \Gamma_1^{(0)}\}$ and $\{\chi_w : w \in \Gamma_2^{(0)}\}$ are rational functions associated to the graphs satisfying:

$$x\varphi_v = \Delta_1\varphi_v \ , \quad v \in \Gamma_1^{(0)} \tag{4.8}$$

$$x\chi_w = \Delta_2\chi_w \ , \quad w \in \Gamma_2^{(0)} \tag{4.9}$$

and $\varphi_{*_1} = \chi_{*_2} = 1$.

Suppose that $\{a_{wv} : w \in \Gamma_2^{(0)},\ v \in \Gamma_1^{(0)}\}$ are non-negative integers satisfying:

(4.10) $\qquad \varphi_v = \sum_w a_{wv}\chi_w$ where $(a_{wv})_w$ lie in $V(\Gamma_2)_+$.

(4.11) $\qquad a_{wv} = 0$ if $d_2(w) > d_1(v)$.

(4.12) $\qquad a_{wv} = 0$ if $d_2(w) - d_1(v) \equiv 1 \bmod 2$.

(4.13) $\qquad (a_{w*_1})_w = *_2.$

Then if the rational functions (χ_w) are linearly independent over $\mathbb{Z}$, $A = [a_{wv}]$ defines a positive linear map of $V(\Gamma_1)$ into $V(\Gamma_2)$ satisfying:

(4.14) $\qquad A$ has no rows or columns zero, and is column finite.

(4.15) $\qquad A$ maps an even (respectively odd) vertex v of $\Gamma_0^{(1)}$ into a linear combination of even (respectively odd) vertices w of $\Gamma_2^{(0)}$ with $d_2(w) \leq d_1(u)$.

(4.16) $\qquad A\Delta_1 = \Delta_2 A.$

(4.17) $\qquad A*_1 = *_2.$

Conversely, if (4.14) - (4.17) hold for some positive linear map $A = [a_{wv}]$ of $V(\Gamma_1)$ into $V(\Gamma_2)$ and there is an unique solution $\{\phi_v\}$ to (4.8), subject to $\phi_{*_1} = 1$, then (4.10) - (4.13) hold.

Proof:

Suppose (4.10) - (4.13) hold. Then from (3.1) we have

$$x\varphi_v = \sum_w a_{wv}\, x\chi_w = \sum_w a_{wv} \sum_u \Delta_2(u,w)\, \chi_u = \sum_u \Big(\sum_w \Delta_2(u,w)\, a_{wv}\Big) \chi_u.$$

Moreover

$$x\varphi_v = \sum_s \Delta_1(s,v)\, \varphi_s = \sum_s \Delta_1(s,v) \sum_u a_{us}\, \chi_u = \sum_u \Big(\sum_s a_{us}\, \Delta_1(s,v)\Big) \chi_u.$$

Thus if (χ_u) are linearly independent we deduce $A\Delta_1 = \Delta_2 A$. Similarly if (4.14) - (4.17) hold, then we have

$$x\Big(\sum_w a_{wv}\, \chi_w\Big) = \sum_w a_{wv} \sum_u \Delta_2(u,w)\, \chi_u = \sum_u \Big(\sum_w \Delta_2(u,w)\, a_{wv}\Big) \chi_u$$

$$= \sum_u \Big(\sum_s a_{us}\, \Delta_1(s,v)\, \chi_u = \sum_s \Delta_1(s,v) \Big(\sum_u a_{us}\, \chi_u\Big).$$

Hence assuming uniqueness of the solution to $x\phi_v = \sum \Delta_1(s,v)\, \phi_s$, we deduce that (4.10) holds. The remainder of the lemma is clear.

Lemma 4.4.

Let Γ_1 and Γ_2 be two connected locally finite graphs with distinguished vertices $*_1$ and $*_2$ respectively.

Then the following conditions are equivalent.

(a) There exists a positive linear map $A : V(\Gamma_1) \to V(\Gamma_2)$ such that

(i) $A\Delta_1 = \Delta_2 A$

(ii) A is column finite (i.e. only a finite number of non-zero entries in each column), and has no rows, or columns zero.

(iii) $A*_1 = *_2$.

(b) There exist unital embeddings $j_n : A(\Gamma_1)_n \to A(\Gamma_2)_n$ such that the following diagrams commute:

$$\begin{tikzcd} A(\Gamma_1)_n \arrow[r, "j_n"] \arrow[d] & A(\Gamma_2)_n \arrow[d] \\ A(\Gamma_1)_{n+1} \arrow[r, "j_{n+1}"] & A(\Gamma_2)_{n+1} \end{tikzcd}$$

and the graph for the embedding j_n is a subgraph of the graph of j_{n+2} obtained by removing the new vertices on level $n + 2$, and all edges adjacent to them.

Proof:

If (b) holds, then we reconstruct the matrix $A = [a_{wv}]$ by letting a_{wv} be the number of edges in the graph of the embedding $j_n : A(\Gamma_1)_n \to A(\Gamma_2)_n$ for any n large enough such that (v,w) are vertices of the graph of j_n. If (a) holds with $A = [a_{wv}]$ we construct for each n a graph j_n with vertices obtained by taking the vertices $(\hat{\Gamma}_1)_n \cup (\hat{\Gamma}_2)_n$ at the n^{th} levels of those of $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$, with a_{wv} edges between w in $(\hat{\Gamma}_2)_n$ and v in $(\hat{\Gamma}_1)_n$. The dimensions of the factors of $A(\Gamma_1)_n$ and $A(\Gamma_2)_n$ are given by the components of $\Delta_1^n *_1$, $\Delta_2^n *_2$ respectively. Since

$A\Delta_1^n *1 = \Delta_2^n *2$, we see that j_n defines a unital embedding of $A(\Gamma_1)_n$ in $A(\Gamma_2)_n$, and as $A\Delta_1 = \Delta_2 A$ the diagram in (b) commutes.

Theorem 4.5.

Let Γ be a connected locally finite bipartite graph with distinguished vertex $*$. Define a_m in $V(\Gamma)$ by

$$a_m = S_m(\Delta)a_0 = \Delta a_{m-1} - a_{m-2}$$

$$a_0 = *, \quad a_{-1} = 0$$

(a) If $\|\Delta\| < 2$, then $\|\Delta\| = 2\cos[\pi/(n+1)]$, $a_0 \gneq 0, \ldots, a_{n-1} \gneq 0$, $a_n = 0$, and there is an inclusion of graph algebras $j : A(A_n) \to A(\Gamma)$ given by the matrix $A = [a_0, \ldots, a_{n-1}]$.

(b) If $\|\Delta\| \geq 2$, then $a_m \gneq 0$ for all m, and there is an inclusion of graph algebras $j: A(A_\infty) \to A(\Gamma)$ given by the matrix $A = [a_0, a_1, a_2, \ldots]$.

Proof:

The formula

$$\Delta a_n = a_{n-1} + a_{n+1}$$

says that $A = [a_0, a_1, \ldots]$ satisfies $\Delta A = A\Delta(A_\infty)$. The result then follows from Lemmas 4.2 and 4.4.

§5. The dimension group of $A(\Gamma)$ for a finite graph Γ.

We now modify the argument of section 3 to calculate the dimension groups of the finite graphs $T_{p,q,r}$, $p \geq q$, $q = 1,2$, $r \geq 2$, (note that $T_{p,1,r} = A_{p+r-1}$).

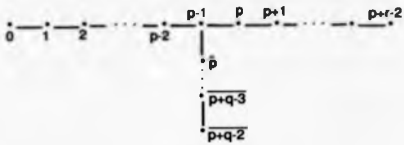

Figure 5 : $T_{p,q,r}$

Let $\{Q_v(t) : v \in T^{(0)}_{p,q,\infty}\}$ be the polynomials associated to the infinite graph $T_{p,q,\infty}$.

As before let

$$R = \operatorname{lin}_{\mathbb{Z}}\{Q_{(v,n)}(t) ; (v,n) \in \hat{T}^{(0)}_{p,q,\infty}\} = \mathbb{Z}[t].$$

Now let I be the ideal in R generated by $Q_{(p+r-1,0)} = Q_{p+r-1}$, and denote by $\bar{R}$ the quotient ring R / I. For $f(t) \in \mathbb{Z}[t]$, we write $\bar{f}$ for its image $f + I$ in R / I. The set $\{1, \bar{t}, \ldots, \bar{t}^{d-1}\}$ forms a basis of $\bar{R}$, considered as a $\mathbb{Z}$-module where $d = \deg Q_{p+r-1}$, and $\bar{R} = \mathbb{Z}[\bar{t}]$.

We associate to each vertex (v,n) of $\hat{T}_{p,q,r}$ the element $\bar{Q}_{(v,n)}$ of the ring $\bar{R}$.

Lemma 5.1.

Consider the graph $T_{p,q,r}$, $p \geq q$, $q = 1,2$, with associated set $\{\bar{Q}_{(v,n)} ; (v,n) \in \hat{T}^{(0)}_{p,q,r}\}$.

(a) The set $\{\bar{Q}_{(v,n)}\}$ satisfies the following splitting rules:

$$\bar{Q}_{(v,n)} = \sum_{w} \bar{\Delta}(v,w)\,\bar{Q}_{(w,n+1)}$$

where $\bar{\Delta}$ is the incidence matrix for the graph $T_{p,q,r}$.

(b) The linear span over $\mathbb{Z}$ of the set $\{\bar{Q}_{(v,n)}\}$ in $\bar{R}$, is the whole of $\bar{R}(=\mathbb{Z}[t])$.

(c) If p is even, then for each fixed k, the elements of $\bar{R}$ associated to level $2k+1$ of $\hat{T}_{p,2,\infty}$ are linearly independent over $\mathbb{Z}$. If p is odd, those associated with level $2k$ are independent over $\mathbb{Z}$.

(d) $\quad (1-\bar{t})\,\bar{Q}_{(v,n)} = \sum'_{i\in P^2(v)} \bar{Q}_{r(i),n+2)}$

where Σ' means we omit the path with endpoint the vertex v exactly once from the summation.

Proof:

(a) and (d) follow immediately from Lemma 3.6.

(b) and (c): First note that if p is even (or $q=1$) then $\deg Q_{p+r-1} = \left[\frac{p+r-1}{2}\right]$ for $r \geq p+1$, and if p is odd then $\deg Q_{p+r-1} = \left[\frac{p+r+1}{2}\right]$ for $r \geq p+1$. Now suppose p is even. If r is even, then $p+r-3 = 2d-1$, where $d = \deg Q_{p+r-1}$, and if r is odd then $p+r-2 = 2d-1$. We know that for $k \leq d-1$ there are $k+1$ vertices on level $2k+1$ of $\hat{T}_{p,q,\infty}$, and hence $\hat{T}_{p,q,r}$. Also the set $\{t^j\}_{j=0}^{k}$ is in the linear span over $\mathbb{Z}$ of the polynomials associated with level $2k+1$. Thus $\{\bar{t}^j\}_{j=0}^{k}$ are in the linear span over $\mathbb{Z}$ of the set $\{\bar{Q}_{(v,2k+1)}\}$ associated to level $2k+1$ of $\hat{T}_{p,q,r}$, for $k \leq d-1$. In particular when r is even (r odd), on level $p+r-3 = 2d-1$ $\quad$ ($p+r-2 = 2d-1$ respectively) there are d vertices, and the linear span over $\mathbb{Z}$ of the set $\{\bar{Q}_{(v,2d-1)}\}$ will contain $\{\bar{t}^j\}_{j=0}^{d-1}$, and so will be

$\mathbb{Z}[\bar{t}]$. The same will be true for levels $2k+1$, for $k \geq d-1$. The case when p is odd is similar.

It follows from the preceding Lemma, using a similar argument to that in section 3, that for $q = 1,2$

$$\bar{R} = \mathrm{lin}_{\mathbb{Z}} \{\bar{Q}_{(v,n)} ; (v,n) \in \hat{T}^{(0)}_{p,q,r}\} = \mathbb{Z}[\bar{t}] \cong K_o(A(T_{p,q,r}))$$

and that the positive cone $K_o(A(T_{p,q,r}))_+$ may be identified with $\bar{R}_+$ where

$$\bar{R}_+ = \mathrm{lin}_{\mathbb{N}} \{\bar{Q}_{(v,n)} ; (v,n) \in \hat{T}^{(0)}_{p,q,r}\}.$$

We now characterise the elements of $\mathbb{Z}[\bar{t}]$ corresponding to the positive cone of $K_o(A(T_{p,q,r}))$.

It is clear that $(\bar{R}, \bar{R}_+)$ is a partially ordered, unperforated abelian group. Also using a similar argument to section 3 one can show that $\bar{1} = \bar{Q}_{(0,0)}$ is an order unit. Also $\bar{t}\,\bar{R}_+ \in \bar{R}_+$, by definition, and $(1-\bar{t})\bar{R}_+ \in \bar{R}_+$, by Lemma 5.1. Then by a slight modification of Lemma 3.7, we see that any extremal state on $(\bar{R}, \bar{R}_+)$ is multiplicative. Hence if ϕ is an extremal state on $(\bar{R}, \bar{R}_+)$, we have $\phi(\bar{t}) = \lambda$, for some $\lambda \in \mathbb{R}$. But $\bar{t} \in \bar{R}_+$, and so $\lambda \geq 0$. Also, since $\bar{Q}_{p+r-1} = \bar{0}$, $Q_{p+r-1}(\lambda) = \phi(\bar{Q}_{p+r-1}) = 0$, and so λ is a root of Q_{p+r-1}. If $\lambda > \gamma_{p+r-1}$, where γ_{p+r-1} is the smallest root of Q_{p+r-1}, we see from the argument of Lemma 3.4, that there is $v \in T^{(0)}_{p,q,r}$ such that $Q_v(\lambda) < 0$. But $Q_v(\lambda) = \phi(\bar{Q}_{(v,0)}) \geq 0$ for all $v \in T^{(0)}_{p,q,r}$.

Hence $\lambda = \gamma_{p+r-1}$. Thus we have shown that $(\bar{R}, \bar{R}_+)$ has a unique extremal state, given by evaluation. Note that by truncation, and telescoping we can consider $(\bar{R}, \bar{R}_+)$ to be the limit of a simple stationary system, which by [E], has a unique extremal state. Thus, in the particular case $\Gamma = T_{p,q,r}$, $q = 1,2$, we have an independent verification of this result.

Now $(\phi_v(\beta_{p+r-1}))_{v \in T^{(0)}_{p,q,r}}$ is a Perron-Frobenius eigenvector for the incidence matrix $\bar{\Delta}$ of $T_{p,q,r}$, and so $\phi_v(\beta_{p+r-1}) > 0$ for each $v \in T^{(0)}_{p,2,r}$. Then it follows from the definition of $Q_v(t)$, that $Q_v(\gamma_{p+r-1}) > 0$ for each $v \in T^{(0)}_{p,2,r}$. Hence the positive cone $\bar{R}_+$ is contained in the set

$$\{\bar{f} \,;\, f \in \mathbb{Z}[t],\ f(\gamma_{p+r-1}) > 0\} \cup \{0\}.$$

Now suppose that $f \neq 0$ is an element of this set, then if ϕ is the extremal state on $(\bar{R},\bar{R}_+)$ we have

$$\phi(\bar{f}) = f(\gamma_{p+r-1}) > 0$$

and so by [GH,G], it follows that $\bar{f} \in \bar{R}_+$.

Theorem 5.2.

For $p \geq q$, $q = 1,2$, $r < \infty$ we have

$$K_o(A(T_{p,q,r})) \cong \mathbb{Z}[t]/I$$

where I is the ideal in $\mathbb{Z}[t]$, generated by Q_{p+r-1}, and $K_o(A(T_{p,q,r}))_+$ corresponds under this isomorphism to the set

$$\{\bar{f} = f + I \,;\, f(\gamma) > 0\} \cup \{0\}$$

where $\gamma = ||T_{p,q,r}||^{-2}$.

Corollary 5.3.

$K_o(A(T_{p,q,r}))$ is a partially ordered ring. In particular there exist non-negative integers $a^{\gamma}_{\alpha\bar{\beta}}$ for $\alpha,\beta,\gamma \in \hat{T}^{(0)}_{p,q,r}$ such that

$$\bar{Q}_{\alpha}\bar{Q}_{\beta} = \Sigma\, a^{\gamma}_{\alpha\beta}\,\bar{Q}_{\gamma} \tag{5.1}$$

In section 4 we constructed embeddings of $A(A_n)$ in $A(\Gamma)$, where n was finite if $\|\Gamma\| < 2$, and $n = \infty$ otherwise. We now examine these embeddings at the dimension group level in the case $\Gamma = T_{p,q,r}$ $(q = 1,2)$.

Recall from section 3 that $K_o(A(A_\infty)) \cong \mathbb{Z}[t]$, and the positive cone can be identified with the set

$$\{Q \in \mathbb{Z}[t]\,;\, Q(\lambda) > 0,\ \lambda \in (0,\tfrac{1}{4}]\} \cup \{0\}.$$

Also, from Theorem 5.2 we have $K_o(A(A_n)) \cong \mathbb{Z}[t]\,/\,I$, where I is the ideal generated by the Jones polynomial $P_n(t)$. The positive cone may be identified with the set

$$\{\bar{Q} = Q + I\,;\, Q(\gamma) > 0\} \cup \{0\}$$

where $\gamma = \|A_n\|^{-2} = (2\cos(\pi/(n+1)))^{-2}$.

Suppose that $\Gamma = T_{p,q,r}$ $(q = 1,2)$, and $\|\Gamma\| \geq 2$. Then since $K_o(A(\Gamma)) \cong \mathbb{Z}[t]/J$, where J is the principal ideal $\langle Q_{p+r-1}\rangle$ in $\mathbb{Z}[t]$, the quotient map onto $\mathbb{Z}[t]\,/\,J$ gives a surjective ring homomorphism $\varphi : K_o(A(A_\infty)) \to K_o(A(\Gamma))$. Note that $\ker\varphi = J$. Since the positive cone of $K_o(A(\Gamma))$ can be identified with the set

$$\{\bar{\bar{Q}} = Q + J\,;\, Q(\gamma) > 0\} \cup \{0\}$$

where $\gamma = \|\Gamma\|^{-2} \in (0,\frac{1}{4}]$, the map φ is positive, and since $f \in \ker\varphi$ implies $f(\gamma) = 0$, $K_o(A(A_\infty))_+ \cap \ker\varphi = \{0\}$, and $\varphi(K_o(A(A_\infty))_+)$ is a subset of $K_o(A(\Gamma))_+$.

Then since $K_o(A(\Gamma))_+$ is generated by the polynomials

$$[\bar{\bar{Q}}_{(v,m)} = Q_{(v,m)} + J \; ; \; (v,m) \in \hat{T}^{(0)}_{p,q,r}]$$

there are non-negative integers $a_{n\alpha}$, $\alpha \in \hat{T}^{(0)}_{p,q,r}$ such that

$$\bar{\bar{P}}_n = \sum_{\alpha} a_{n\alpha} \bar{\bar{Q}}_{\alpha}, \tag{5.2}$$

for $n = 0,1,2,\ldots$.

Now suppose that $\|\Gamma\| < 2$, so $\Gamma = A_k, D_k, E_6, E_7, E_8$ as in Figure 6, and $\|\Gamma\| = 2\cos(\pi/(n+1))$ for some $n \geq 2$.

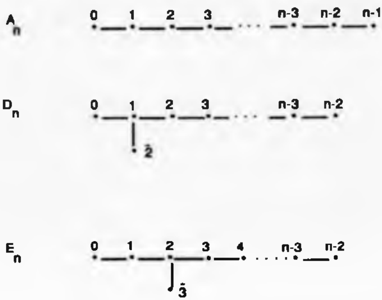

Figure 6

Now $K_0(A(\Gamma)) \simeq \mathbb{Z}[t] / J$ where J is the ideal generated by Q_{p+r-1}. One can check, by comparing Coxeter exponents of the graphs A_n, and Γ, that $Q_{p+r-1} \mid P_n$, and hence $J = (Q_{p+r-1}) \supseteq (P_n) = I$. Thus the map

$$\phi : \mathbb{Z}[t] / I \to \mathbb{Z}[t] / J$$

given by $\phi(\bar{f}) = \bar{\bar{f}}$ is a surjective ring homomorphism, and its kernel is the ideal in $\mathbb{Z}[t]/I$ generated by $\bar{Q}_{p+r-1}$. And so from the identification of the positive cones, we have a positive map

$$\phi : K_o(A(A_n)) \to K_o(A(\Gamma))$$

such that

$$K_o(A(A_n))_+ \cap \ker \phi = \{0\},$$

and

$$\phi(K_o(A(A_n))_+) = K_o(A(\Gamma))_+.$$

Hence there are non-negative integers $a_{m\alpha}$, $\alpha \in \hat{T}^{(0)}_{p,q,r}$, such that

$$\bar{\bar{P}}_m = \sum_\alpha a_{m\alpha} \bar{\bar{Q}}_\alpha \tag{5.3}$$

for $m = 0,1,\ldots,n-1$.

That these maps between dimension groups can be lifted to the algebra level is a consequence of Theorem 4.5. In fact one can obtain expansions (5.2) and (5.3) of $\bar{\bar{P}}_m$ by taking the vectors $a_m = (a_{m\alpha})_\alpha \in V(\Gamma)$ as coefficients. Note also that the $\bar{\bar{Q}}_\alpha$ that occur in this expansion with non-zero coefficients are associated to level m of $\hat{T}_{p,q,r}$. We illustrate this in Figures 7 - 10 for the cases $\Gamma = E_6, E_7, E_8, E_9 = T_{3,2,6}$. The graphs E_6, E_7, E_8 have norm $2\cos\frac{\pi}{m}$, with $m = 12, 18, 30$ respectively. Thus we have embeddings of $A(A_{11}), A(A_{17}), A(A_{29})$ in $A(E_6)$, $A(E_7)$, and $A(E_8)$ respectively. The graph E_9 has norm 2, and so we have an embedding of $A(A_\infty)$ in $A(E_9)$.

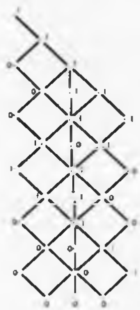

Figure 7 $A(A_{11}) \subset A(E_6)$

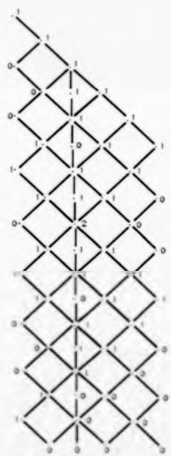

Figure 8 $A(A_{17}) \subset A(E_7)$

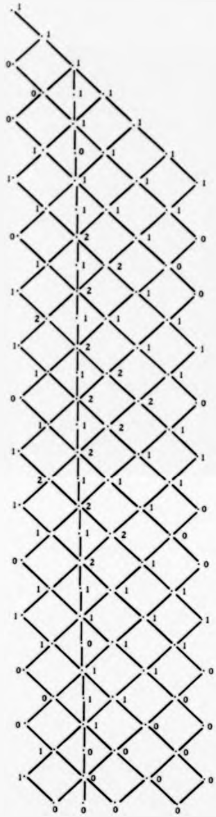

Figure 9 $A(A_{29}) \subseteq A(E_8)$

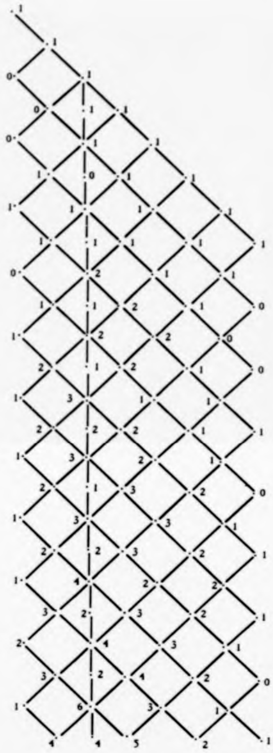

Figure 10 : $A(A_\infty) \subseteq A(E_9)$

§6. Fusion Rules.

In a rational conformal field theory [V,MS1,MS2] with primary fields $\{\phi_i\}$ we have the formal product rule, or operator product expansion

$$\phi_j \times \phi_k = \Sigma\, N^i_{jk}\, \phi_i \tag{6.1}$$

where the non-negative integers N^i_{jk} are known as the fusion rules. Since this product is associative, the algebra generated by the $\{\phi_i\}$ with this product will have a representation as linear operators. In fact the matrices $N_j = (N^i_{jk})_{k,i}$ will afford such a representation, i.e.

$$N_j N_k = \Sigma\, N^i_{jk}\, N_i \tag{6.2}.$$

The matrices N_j are symmetric, mutually commuting, non-negative, and integral. Thus they will generate a commutative ring of symmetric matrices over $\mathbb{Z}$, which when equipped with the positive cone

$$\mathrm{lin}_{\mathbb{N}}\, \{N_i\} \tag{6.3}$$

will be a partially ordered subring of $M_n(\mathbb{Z})$ where n is the number of primary fields.

We will now show that for certain graphs Γ, one can use the rational functions $\{\phi_v ; v \in \Gamma^{(0)}\}$ defined in section 3 to construct such families of matrices, and that the ring they generate is $\mathbb{Z}[\Delta]$, where Δ is the incidence matrix of Γ. It is then an easy matter to find a unitary matrix S that diagonalises the fusion rules [V,MS1,MS2].

Example 6.1.

Let $\Gamma = A_n$, then the incidence matrix is given by

$$\begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & \ddots & & \\ & & \ddots & \ddots & & \\ & & & & \ddots & 1 \\ & & & & 1 & 0 \end{pmatrix} \tag{6.4}$$

Now since $||\Gamma|| = 2\cos(\pi/n+1)$, we know by Theorem 4.5 that the matrices $S_j(\Delta)$, $j = 0,1,\dots,n-1$, are non-negative and non-zero. It is clear also that they are symmetric and mutually commute. It follows immediately that they satisfy the formal product rule

$$S_j(\Delta)S_k(\Delta) = \sum_{i=0}^{n-1} N_{jk}^{i}\, S_i(\Delta) \tag{6.5}$$

where $S_j(\Delta) = (N_{jk}^{i})_{k,i=0}^{n-1}$. These are, in fact, the same rules that occur in the Wess-Zumino-Witten $SU(2)_{n-1}$ model [GW]. Since the characteristic polynomial for Δ is $S_n(x)$, it is easy to see that the ring generated over $\mathbb{Z}$ by the $\{S_j(\Delta)\}$ is $\mathbb{Z}[\Delta]$. Now $S_n(x)$ has distinct zeros $\beta_k = 2\cos(k\pi/(n+1))$, $k = 1,2,\dots,n$. Thus these are the eigenvalues for Δ, and the corresponding eigenvectors are given by

$$x_k = (S_o(\beta_k), S_1(\beta_k), \dots, S_{n-1}(\beta_k))^T$$

for $k = 1,2,\dots,n$. Thus the matrix

$$S = \left[\frac{x_1}{||x_1||}, \frac{x_2}{||x_2||}, \dots, \frac{x_n}{||x_n||}\right] \tag{6.6}$$

is unitary, and diagonalises Δ, i.e.

$$S^*\Delta S = \mathrm{diag}(\beta_1, \beta_2, \dots, \beta_n).$$

The x_k will also be eigenvectors of $S_j(\Delta)$, and so

$$S_j(\Delta)x_k = \lambda_j^{(k)} x_k \tag{6.7}$$

where $\lambda_j^{(k)} = S_j(\beta_k)$, for $j = 0,\dots,n-1$, $k = 1,\dots,n$. Thus the same matrix S will diagonalise each $S_j(\Delta)$, and

$$S^*S_j(\Delta)S = \mathrm{diag}\,(S_j(\beta_1), S_j(\beta_2), \ldots, S_j(\beta_n)) \tag{6.8}$$

Note also that by (6.5) the eigenvalues $\lambda_j^{(k)}$ satisfy

$$\lambda_i^{(k)} \lambda_j^{(k)} = \Sigma\, N_{ij}^{\ell} \lambda_{\ell}^{(k)} \tag{6.9}$$

for $k = 1,\ldots,n$, and thus determine the n one-dimensional representations of the fusion rule algebra. Now using the fact that

$$S_n(z + z^{-1}) = z^n + z^{n-2} + \ldots + z^{-(n-2)} + z^{-n} \tag{6.10}$$

and that

$$\beta_k = 2\cos(k\pi/(n+1)) = \exp(ik\pi/(n+1)) + \exp(-ik\pi/(n+1)) \tag{6.11}$$

it is easy to show that

$$\lambda_j^{(k)} = S_j(\beta_k) = \frac{\sin\left(\frac{(j+1)k\pi}{n+1}\right)}{\sin\left(\frac{k\pi}{n+1}\right)} \tag{6.12}$$

and

$$S_{jk} = \frac{S_j(\beta_k)}{\| x_k \|} = \left(\frac{2}{n+1}\right)^{\frac{1}{2}} \sin\left(\frac{(j+1)k}{n+1}\pi\right) \tag{6.13}$$

for $j = 0,1,\ldots,n-1$, $k = 1,\ldots,n$. Thus the matrix $S = (S_{jk})$ is symmetric, and since it is also unitary, it satisfies the relation $S^2 = I$.

This is the same matrix S as appears in [GW, V], to describe the action of the generator $S : \tau \to -1/\tau$ of the modular group $PSL(2,\mathbb{Z})$ on the space of characters of the conformal field theory model.

Example 6.2.

Let $\Gamma = E_6$, then the incidence matrix Δ is

$$\begin{pmatrix} 0&1&0&0&0&0\\ 1&0&1&0&0&0\\ 0&1&0&1&0&1\\ 0&0&1&0&1&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0 \end{pmatrix}$$

where we have ordered the vertices as $0,1,2,3,4,\bar{3}$ (see Figure 6). The characteristic polynomial for Δ is $x^6 - 5x^4 + 5x^2 - 1$, hence Δ is invertible. Recall from section 3 the rational functions associated to the graph E_∞. For the subgraph E_6 they are

$$\phi_o = 1 \quad \phi_1 = x \quad \phi_2 = x^2 - 1$$

$$\phi_3 = x^3 - 3x + x^{-1} \qquad \phi_4 = x^4 - 4x^2 + 2 \qquad \phi_{\bar{3}} = x - x^{-1}.$$

Now since Δ satisfies its own characteristic polynomial we have

$\Delta^{-1} = \Delta^5 - 5\Delta^3 + 5\Delta$. Thus the matrices $\phi_v(\Delta)$, $v \in E_6^{(0)}$ are contained in, and actually generate $\mathbb{Z}[\Delta]$. They are given below:

$$\phi_o(\Delta) = 1 \qquad \phi_1(\Delta) = \Delta$$

$$\phi_2(\Delta) = \begin{pmatrix} 0&0&1&0&0&0\\ 0&1&0&1&0&1\\ 1&0&2&0&1&0\\ 0&1&0&1&0&1\\ 0&0&1&0&0&0\\ 0&1&0&1&0&0 \end{pmatrix} \qquad \phi_3(\Delta) = \begin{pmatrix} 0&0&0&1&0&0\\ 0&0&1&0&1&0\\ 0&1&0&1&0&1\\ 1&0&1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0 \end{pmatrix}$$

$$\phi_4(\Delta) = \begin{pmatrix} 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&1 \end{pmatrix} \qquad \phi_{\bar{3}}(\Delta) = \begin{pmatrix} 0&0&0&0&0&1\\ 0&0&1&0&0&0\\ 0&1&0&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&1\\ 1&0&0&0&1&0 \end{pmatrix}$$

Thus they constitute a family of *non-negative*, symmetric, mutually commuting integral matrices. They satisfy the product rule

$$\phi_v(\Delta)\,\phi_w(\Delta) = \Sigma\, N^u_{vw}\,\phi_u(\Delta)$$

where $\phi_v(\Delta) = (N^u_{vw})$.

The eigenvalues of Δ are $\beta_k = 2\cos(k\pi/12)$, $k = 1,4,5,7,8,11$, so $\beta_1 = -\beta_{11} = \sqrt{(2+\sqrt{3})}$, $\beta_4 = -\beta_8 = 1$, and $\beta_5 = -\beta_7 = \sqrt{(2-\sqrt{3})}$. The corresponding eigenvectors are

$$x_k = (\phi_o(\beta_k), \phi_1(\beta_k), \phi_2(\beta_k), \dots, \phi_{\bar{3}}(\beta_k))^T.$$

The eigenvalues are distinct, so the matrix

$$S = \left[\frac{x_1}{\|x_1\|}, \frac{x_4}{\|x_4\|}, \frac{x_5}{\|x_5\|}, \frac{x_7}{\|x_7\|}, \frac{x_8}{\|x_8\|}, \frac{x_{11}}{\|x_{11}\|}\right]$$

is unitary and

$$S^*\Delta S = \mathrm{diag}\,(\beta_1, \beta_4, \beta_5, \beta_7, \beta_8, \beta_{11}).$$

One also has

$$\phi_v(\Delta)\,x_k = \lambda_v^{(k)}\,x_k$$

where $\lambda_v^{(k)} = \phi_v(\beta_k)$, $v \in E_6^{(0)}$, $k = 1,4,5,7,8,11$, and

$$S^*\phi_v(\Delta)S = \mathrm{diag}\,(\phi_v(\beta_1), \dots, \phi_v(\beta_{11})).$$

§7. Commuting Squares.

Suppose that Γ_1 and Γ_2 are finite connected graphs, with distinguished vertices $*_1$ and $*_2$, and incidence matrices Δ_1 and Δ_2. Let v and w denote the Perron-Frobenius eigenvectors of Γ_1 and Γ_2 respectively, normalized so that $v_{*_1} = w_{*_2} = 1$. Let Tr_1 and Tr_2 be the unique positive traces on $A(\Gamma_1)$ and $A(\Gamma_2)$ respectively [E]. Suppose also that Γ_1 and Γ_2 satisfy the conditions of Lemma 4.4, with $j : A(\Gamma_1) \to A(\Gamma_2)$, and $A = (a_{ij}) : V(\Gamma_1) \to V(\Gamma_2)$. Let π be the G.N.S. representation of $A(\Gamma_2)$ with respect to Tr_2. Then the weak closure $M(\Gamma_2)$ of $\pi(A(\Gamma_2))$ is isomorphic to the hyperfinite type II_1 factor, and we can identify $M(\Gamma_1)$ with the weak closure of $\pi(j(A(\Gamma_1)))$, a subfactor of $M(\Gamma_2)$.

In section 4.5 of [GHJ] irreducible subfactors of the hyperfinite type II_1-factor are constructed from A-D-E graphs. The construction involves, using our notation, a unital embedding $\phi : A(A_m) \to A(\Gamma)$, with $||\Gamma|| = 2\cos(\pi/(m+1))$. We will show in Proposition 7.4 that the embedding is given by a positive linear map $A : V(A_m) \to V(\Gamma)$ intertwining the incidence matrices of A_m, and Γ, and is identical to that constructed in Theorem 4.5(a). First we show that any embedding constructed via an intertwining matrix as in Theorem 4.5(a) is in fact trace preserving. We also obtain a simple formula for the index based on [PP], and give a proof of a criterion for irreducibility which is stated without proof in [HS].

Proposition 7.1.

Suppose that Γ_1, and Γ_2 are finite connected graphs with distinguished vertices $*_1$, and $*_2$. Suppose also that there exists a positive linear map $A : V(\Gamma_1) \to V(\Gamma_2)$ such that

(a) $A\Delta_1 = \Delta_2 A$

(b) A has no rows, or columns zero

(c) $A*_1 = *_2$.

Then the unital embedding, $j : A(\Gamma_1) \to A(\Gamma_2)$, given by Lemma 4.4, is trace preserving, i.e.

$$Tr_2 \circ j = Tr_1.$$

Proof:

Put $y = A^T w$. Then since A^T has no rows zero, y must have strictly positive entries. Moreover,

$$\Delta_1 y = \Delta_1 A^T w = A^T \Delta_2 w = ||\Gamma_2|| A^T w = ||\Gamma_2|| y,$$

and so y is a Perron-Frobenius eigenvector for Δ_1. Also, since $A*_1 = *_2$, we have $y*_1 = w*_1 = 1$. Hence $A^T w = y = v$. Now suppose that p_i is a minimal projection in $A(\Gamma_1)_n$ corresponding to vertex (i,n) of $\hat{\Gamma}_1$ Then there are minimal projections q_j corresponding to vertices (j,n) of $\hat{\Gamma}_2$ such that

$$j_n(p_i) = \sum_j a_{ji} q_j$$

where $j_n : A(\Gamma_1)_n \to A(\Gamma_2)_n$. But then,

$$Tr_2(j_n(p_i)) = \sum_j a_{ji} Tr_2(q_j) = \sum_j a_{ji} \beta^{-n} w_j = \beta^{-n} (A^T w)_i = \beta^{-n} v_i = Tr_1(p_i).$$

Remark 7.2.

It follows from the proof of the preceding proposition, that $||\Gamma_2||$ is the Perron-Frobenius eigenvalue of Δ_1. Hence $||\Gamma_2|| = ||\Gamma_1||$, by uniqueness.

Proposition 7.3.

Suppose that Γ_1 and Γ_2 are finite connected graphs, with distinguished vertices $*_1$ and $*_2$. Suppose also that there exists a positive linear map $A : V(\Gamma_1) \to V(\Gamma_2)$ such that

(a) $A\Delta_1 = \Delta_2 A$

(b) A has no rows, or columns zero

(c) $A*_1 = *_2$.

Then the index of $M(\Gamma_1)$ in $M(\Gamma_2)$ satisfies the inequality

$$[M(\Gamma_2) : M(\Gamma_1)] \leq (Av)_{*_2}.$$

Moreover, if the diagram

$$\begin{tikzcd} A(\Gamma_1)_n \arrow[r, "j_n"] \arrow[d] & A(\Gamma_2)_n \arrow[d] \\ A(\Gamma_1)_{n+1} \arrow[r, "j_{n+1}"] & A(\Gamma_2)_{n+1} \end{tikzcd}$$

is a commuting square [GHJ], for all n, then

$$[M(\Gamma_2) : M(\Gamma_1)] = (Av)_{*_2}.$$

If $[M(\Gamma_2) : M(\Gamma_1)]$ is finite, then $M(\Gamma_1)$ is an irreducible subfactor of $M(\Gamma_2)$, [HS].

Proof:

According to Proposition 2.6 of [PP], the index satisfies

$$[M(\Gamma_2) : M(\Gamma_1)]^{-1} \geq \lim_n \sup \lambda[A(\Gamma_2)_n, A(\Gamma_1)_n],$$

and if the commuting squares condition is satisfied, then

$$[M(\Gamma_2) : M(\Gamma_1)]^{-1} = \lim_n \lambda[A(\Gamma_2)_n, A(\Gamma_1)_n].$$

Here, by Theorem 6.1 of [PP], $\lambda[A(\Gamma_2)_n, A(\Gamma_1)_n]$ is given by

$$\lambda[A(\Gamma_2)_n, A(\Gamma_1)_n]^{-1} = \max_{\ell} \left(\sum_k b^n_{k\ell} \frac{\beta^{-n} v_k}{\beta^{-n} w_l} \right)$$

where

$$b^n_{k\ell} = \min (a_{\ell k}, (\Delta_1^n *_1)_k).$$

Thus for n sufficiently large, $b^n_{k\ell} = a_{\ell k}$, and so

$$\lambda[A(\Gamma_2)_n, A(\Gamma_1)_n]^{-1} = \max_{\ell}\left(\sum_k a_{\ell k}\frac{v_k}{w_\ell}\right) = \max_{\ell}\left(\frac{(Av)_\ell}{w_\ell}\right)$$

But A has no rows of zero, so $(Av)_\ell > 0$ for all $\ell \in \Gamma_2^{(0)}$. Hence Av is a Perron Frobenius eigenvector for Δ_2 (since $A\Delta_1 = \Delta_2 A$), and so $Av = \mu w$, for some $\mu > 0$. Thus for n sufficiently large

$$\lambda[A(\Gamma_2)_n, A(\Gamma_1)_n]^{-1} = \mu$$

where we can take $\mu = \dfrac{(Av)_{*_2}}{w_{*_1}} = (Av)_{*_2}$.

Let p denote the minimal central projection in the simple component of $A(\Gamma_1)_n$ corresponding to the vertex $*_1$ of Γ_1. Then $p \in A(\Gamma_1)'_n \cap A(\Gamma_2)_n$, which is semisimple, and decomposes as a direct sum of simple components in one to one correspondence with pairs of vertices (v,w), with v, and w labeling simple components of $A(\Gamma_1)_n$, and $A(\Gamma_2)_n$ respectively. The dimension of a component labeled by a pair (v,w) being $(a_{wv})^2$. Since $A*_1 = *_2$, the projection p can be identified with the central projection in the one dimensional component labeled by the pair $(*1,*2)$. Thus $p(A(\Gamma_1)'_n \cap A(\Gamma_2)_n)p = \mathbb{C}p$, for all even n. But by [Wen2]

$$\dim(M(\Gamma_1)' \cap M(\Gamma_2)) \le \dim(p(A(\Gamma_1)'_n \cap A(\Gamma_2)_n)p),$$

for sufficiently large n, and so irreducibility follows.

Proposition 7.4.

Suppose that Γ_1 and Γ_2 are locally finite connected graphs with distinguished vertices $*_1$ and $*_2$ respectively. Let Tr_1, and Tr_2, be Markov traces on $A(\Gamma_1)$, and $A(\Gamma_2)$ respectively. Let $(e_n)_{n\in\mathbb{N}}$, $(f_n)_{n\in\mathbb{N}}$ denote families of Jones projections in $A(\Gamma_1)$, and $A(\Gamma_2)$ respectively, and $\phi : A(\Gamma_1) \to A(\Gamma_2)$ a unital embedding such that

(a) The diagram:

$$\begin{tikzcd} A(\Gamma_1)_n \arrow[r, "\phi_n"] \arrow[d, "k_n"'] & A(\Gamma_2)_n \arrow[d, "h_n"] \\ A(\Gamma_1)_{n+1} \arrow[r, "\phi_{n+1}"'] & A(\Gamma_2)_{n+1} \end{tikzcd}$$

commutes for all n, where $\phi_n = \phi \mid A(\Gamma_1)_n$, and h_n, k_n are $*$-homomorphisms.

(b) $Tr_2 \circ \phi_n = Tr_1$.

(c) $e_n \in A(\Gamma_1)'_{n-1} \cap A(\Gamma_1)_{n+1}$; $\quad f_n \in A(\Gamma_2)'_{n-1} \cap A(\Gamma_2)_{n+1}$.

(d) $\phi(e_n) = f_n$ for all $n \geq 1$, (so $\phi_{n+1}(e_n) = f_n$).

Then there exists a positive linear map $A : V(\Gamma_1) \to V(\Gamma_2)$ such that

1) $A\Delta_1 = \Delta_2 A$

2) A has no rows, or columns zero, and is column finite.

3) $A{*}_1 = {*}_2$.

Proof:

Let p_i^n denote a minimal projection in $A(\Gamma_1)_n$ corresponding to vertex (i,n) of $\hat{\Gamma}_1$. Then $\phi_n(p_i^n)$ is a projection in $A(\Gamma_2)_n$, and so there are minimal projections $q_j^n \in A(\Gamma_2)_n$ corresponding to vertices (j,n) in $\hat{\Gamma}_2$, and non-negative integers $\{b_{ij}\}_j$ such that

$$\phi_n(p_i^n) = \sum_j b_{ji}\, q_j^n \qquad (7.2).$$

The numbers $\{b_{ij}\}_i$ are not all zero, since ϕ_n is injective. Now multiplying (7.2) by f_{n+1}, we have

$$f_{n+1}\phi_n(p_i^n) = \sum_j b_{ji}\, f_{n+1}\, q_j^n,$$

but by (a), and (b)

$$f_{n+1}\phi_n(p_i^n) = \phi_{n+2}(e_{n+1})\,\phi_n(p_i^n) = \phi_{n+2}(e_{n+1}\,p_i^n)$$

and so we have

$$\phi_{n+2}(e_{n+1}\,p_i^n) = \Sigma\, b_{ji}\, f_{n+1}\, q_j^n \qquad (7.3).$$

Then by the Markov property of Tr_1 and Tr_2, and by (iii) we have

$$Tr_1(e_{n+1}p_i) = \beta^{-2}Tr_1(p_i)$$

$$Tr_2(f_{n+1}q_j) = \beta^{-2}Tr_2(q_j)$$

where $\beta = ||\Gamma_1|| = ||\Gamma_2||$. Hence $e_{n+1}p_i$ is a minimal projection in $A(\Gamma_1)_{n+2}$ corresponding to vertex $(i,n+2)$ of $\hat{\Gamma}_1$, and $f_{n+1}q_j$ is a minimal projection in $A(\Gamma_2)_{n+2}$ corresponding to vertex $(j,n+2)$ of $\hat{\Gamma}_2$. It follows from (7.2) and (7.3) that the coefficients occurring in the decomposition of a minimal projection as in (7.2) corresponding to vertex (i,n) of $\hat{\Gamma}_1$, $n \geq 1$, is independent of the level n.

Now put $A = (b_{j,i})_{i\in\Gamma_1^{(0)},\, j\in\Gamma_2^{(0)}}$, then since $A(\Gamma_1)_o \simeq \mathbb{C} \simeq A(\Gamma_2)_o$, and $\phi_o: A(\Gamma_1)_o \to A(\Gamma_2)_o$ we see that $A*_1 = *_2$. Note that since ϕ is unital, the rows of A are non-zero. We now show that $A\Delta_1 = \Delta_2 A$. Let $\Delta_k = (\Delta_{ij}^{(k)})$, $k = 1,2$, then

$$k_n(p_i^n) = \sum_\ell \Delta_{i\ell}^{(1)}\, p_\ell^{n+1}, \quad \text{and} \quad h_n(q_j^n) = \Sigma\, \Delta_{j,m}^{(2)}\, q_m^{n+1}.$$

Then

$$\phi_{n+1}\circ k_n(p_i^n) = \phi_{n+1}\left(\sum_\ell \Delta_{i,\ell}^{(1)}\, p_\ell^{n+1}\right) = \sum_\ell \Delta_{i\ell}^{(1)}\,\phi_{n+1}(p_\ell^{n+1})$$

$$= \sum_\ell \Delta_{i\ell}^{(1)}\left(\sum_k b_{k\ell}\, q_k^{n+1}\right) = \sum_k\left(\sum_\ell b_{k\ell}\,\Delta_{i\ell}^{(1)}\right) q_k^{n+1},$$

and

$$h_n\circ\phi_n(p_i^n) = h_n\left(\sum_j b_{ji}\, q_j^n\right) = \sum_j b_{ji}\, h_n(q_j^n)$$

$$= \sum_j b_{ji} \left(\sum_m \Delta^{(2)}_{j,m} q^{n+1}_m \right) = \sum_m \left(\sum_j b_{ji} \Delta^{(2)}_{j,m} \right) q^{n+1}_m .$$

But $\phi_{n+1} \circ k_n = h_n \circ \phi_n$, hence

$$\sum_j b_{ji} \Delta^{(2)}_{j,m} = \sum_\ell b_{m\ell} \Delta^{(1)}_{i\ell},$$

and $\Delta^{(k)}_{ij} = \Delta^{(k)}_{ji}$, for $k = 1,2$, so

$$\sum_j \Delta^{(2)}_{mj} b_{ji} = \sum_\ell b_{m\ell} \Delta^{(1)}_{\ell i}$$

hence $A\Delta_1 = \Delta_2 A$.

Proposition 7.5.

Suppose that Γ is a locally finite connected graph, with distinguished vertex $*$. Let Tr be a Markov trace on $A(\Gamma)$, and let $\{f_n\}_{n \in \mathbb{N}}$ be the canonical family of Jones projections in $A(\Gamma)$. Then for m as in Theorem 4.5,we can identify $A(A_m)$ with the algebra generated by $\{1, f_1, f_2, \ldots\}$. Let $\phi : A(A_m) \to A(\Gamma)$, be the inclusion map, given by $\phi(1) = 1$, $\phi(f_n) = f_n$. Then there exists a positive linear map $A : V(A_m) \to V(\Gamma)$ such that

(a) $A\Delta(A_m) = \Delta(\Gamma)A$.

(b) A has no rows, or columns zero, and is column finite.

(c) $A*_1 = *$, where $*_1 = 0$ is the distinguished vertex of A_m.

Moreover A is the same matrix as in Theorem 4.5(a). Also the diagrams

$$\begin{array}{ccc} A(A_m)_n & \longrightarrow & A(\Gamma)_n \\ \downarrow & & \downarrow \\ A(A_m)_{n+1} & \longrightarrow & A(\Gamma)_{n+1} \end{array} \qquad (7.4)$$

are commuting squares for all n.

Proof:

Now $\phi : A(A_m) \to A(\Gamma)$ defined by $\phi(1) = 1$, $\phi(f_n) = f_n$ is a unital embedding which clearly satisfies the condition of Proposition 7.4 with $*_1 = 0$, and $*$ varying over Γ. Hence there exists $A = [a_o, a_1, \ldots, a_{m-1}]$ with the required properties.

Now

$$A\Delta(A_m) = [a_1, a_o + a_2, \ldots, a_{m-2}]$$

Thus $\Delta(\Gamma)A = A\Delta(A_m)$ implies that

$$a_1 = \Delta(\Gamma)a_o, \quad a_{m-2} = \Delta(\Gamma)a_{m-1}$$

$$a_{k-1} + a_{k+1} = \Delta(\Gamma)a_k$$

for $k = 1, \ldots, m-2$. Hence A is the same matrix as in Theorem 4.5(a).

Let $E_{A(\Gamma)_n}$ denote the trace preserving conditional expectation of $A(\Gamma)_{n+1}$ onto $A(\Gamma)_n$. Then to show that (7.4) is a commuting square, it is enough to show that

$$E_{A(\Gamma)_n}(A(A_m)_{n+1}) = A(A_m)_n \qquad (7.5).$$

But for $y \in A(\Gamma)_n$, we have

$$Tr(E_{A(\Gamma)_n}(f_n)y) = Tr(f_n y) = \beta^{-2} Tr(y),$$

and so by the positivity of the trace Tr it follows that

$$E_{A(\Gamma)_n}(f_n) = \beta^{-2} 1 \qquad (7.6).$$

Now let $x \in A(A_m)_{n+1}$, then by [J] we can write

$$x = a + \sum b_i f_n c_i \qquad (7.7),$$

where $a, b_i, c_i \in A(A_m)_n$, and the sum is finite. Then by (7.6), and (7.7) we have

$$E_{A(\Gamma)_n}(x) = a + \sum \beta^{-2} b_i c_i ,$$

and (7.5) follows.

It follows from proposition 7.5 that the embedding $A(A_m) \cong A(\tau) \subseteq A(\Gamma)$, for Γ a Dynkin diagram of type A-D-E as in [GHJ] is given by a positive linear map $A : V(A_m) \to V(\Gamma)$, with $||\Gamma|| = 2\cos(\pi/(m+1))$. Then by Proposition 7.2 the index for the corresponding pair of factors is given by

$$[M(\Gamma) : M(A_m)] = (Av)_*$$

where v is the normalized Perron-Frobenius eigenvector for A_m. In addition $M(A_m)$ is an irreducible subfactor of $M(\Gamma)$. The construction given here is different to that in [GHJ].

Example 7.6.

Let $\Gamma = E_6$, with $* = 0$. Then $||\Gamma|| = 2\cos\frac{\pi}{12} = \sqrt{(2+\sqrt{3})}$, and so we have an embedding of $A(A_{11}) \to A(E_6)$. The matrix A may be obtained from Figure 7 and

$$v = (S_i(\beta))_{i=0}^{10}$$

where $\beta = 2\cos(\frac{\pi}{12})$, and S_i is the i^{th} Chebyshev polynomial. Thus

$$[M(E_6) : M(A_{11})] = (Av)_* = S_0(\beta) + S_6(\beta) = 3 + \sqrt{3}$$

§8. An algebraic presentation and matrix units for $A(T_{p,2,r})$.

Consider the graph $T_{p,2,r}$ as in Figures 1 and 5 where $2 \le p < \infty$, $2 \le r \le \infty$. We have already noted in section 2 that $A(\tau) \subset A(T_{p,2,r})$ where $\tau^{-1} = \|T_{p,2,r}\|^2$ if $r < \infty$, and $\tau^{-1} \ge \|T_{p,2,\infty}\|^2$ otherwise. In the path algebra $A(T_{p,2,r})$, the projection e_n may be described as follows. In the notation of section 2,

(8.1) $$A[n-1,n+1] \supset \bigoplus_v \operatorname{End} \ell^2(\iota(v))$$

where the summation is over all even (respectively odd) vertices $v \in T^{(0)}_{p,2,r}$ with $(v,n-1) \in \hat{T}^{(0)}_{p,2,r}$ when n is odd (respectively even). Three situations arise:

(8.2) $\operatorname{End} \ell^2(\iota(v)) = \mathbb{C}$ if $v = 0$, or $v = p + r - 2$, when $r < \infty$, or $v = \bar{p}$.

(8.3) $\operatorname{End} \ell^2(\iota(v)) = M_3$, if $v = p - 1$.

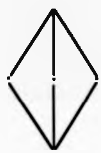

Figure 11

(8.4) $\operatorname{End} \ell^2(\iota(v)) = M_2$ otherwise.

In the identifications of (8.3) and (8.4), we will order paths from left to right. In the first case (8.2), e_n will 1 on these components and in the second and third cases will be the rank one projections in those components given by

$$\frac{1}{\beta\phi_{p-1}}\begin{bmatrix} \phi_{p-2} & (\phi_{p-2}\,\phi_{\bar{p}})^{\frac{1}{2}} & (\phi_{p-2}\,\phi_p)^{\frac{1}{2}} \\ (\phi_{\bar{p}}\,\phi_{p-2})^{\frac{1}{2}} & \phi_{\bar{p}} & (\phi_{\bar{p}}\,\phi_p)^{\frac{1}{2}} \\ (\phi_p\,\phi_{p-2})^{\frac{1}{2}} & (\phi_p\,\phi_{\bar{p}})^{\frac{1}{2}} & \phi_p \end{bmatrix} \tag{8.6}$$

$$\frac{1}{\beta\phi_\eta}\begin{bmatrix} \phi_{\eta-1} & (\phi_{\eta-1}\,\phi_{\eta+1})^{\frac{1}{2}} \\ (\phi_{\eta-1}\,\phi_{\eta+1})^{\frac{1}{2}} & \phi_{\eta+1} \end{bmatrix} \tag{8.7}$$

respectively. We now introduce a new projection, $e_{\bar{p}} \in A[p-1, p+1]$, which lives in End $\ell^2(\iota(v))$, where $v = p - 1$ and is given by projection on the rank one operator corresponding to the middle path, namely $(p-1, \bar{p}, p-1)$ in $T_{p,2,r}$ or $((p-1, p-1), (\bar{p}, p), (p-1, p+1))$ in $\hat{T}_{p,2,r}$:

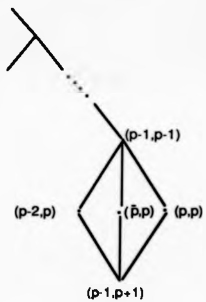

Figure 12

i.e. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ in End $(\ell^2(p-1))$.

We observe:

(8.7) The projection $e_1e_3e_5 \dots e_{2n+1}$ corresponds to the projection $f_{\delta,\delta}$ given by the extreme left hand path δ as in Figure 13.

Figure 13

(8.8) The projection $f_n = 1 - e_1 v \dots v e_{n-1}$ corresponds to the projection $f_{\delta,\delta}$ given by the extreme right hand path δ in Figure 14, for $n \leq p - 1$.

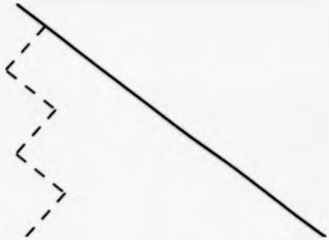

Figure 14

We know by Wenzl [Wen1] see also [GHJ] that if $e_1, \ldots, e_N$ is a sequence of projections satisfying (1.1) and if $s = \left[\frac{N+4}{2}\right]$, then

either (8.9) $\beta = 4\cos^2(\pi/q)$ for some integer q with $3 \le q \le s$

or (8.10) $\beta \ge 4\cos^2(\pi/s)$.

In which case

(8.11) $$f_o = f_1 = 1;$$

(8.12) $$f_{i+1} = f_i - (\beta S_{i-1}/S_i)\, f_i\, e_i\, f_i$$

where $S_i = S_i(\beta)$, for $i = 1,2, \ldots, N-2$, for details see below (8.30)-(8.31). We can then easily verify (8.8) from (8.11)-(8.12). Moreover, we can then deduce the following relations:

(8.13) $$e_n\, e_{\bar{p}} = 0, \quad n = 1,2, \ldots, p-1$$

(8.14) $$e_n\, e_{\bar{p}} = e_{\bar{p}}\, e_n, \quad n = p+1, p+2, \ldots\ldots$$

(8.15) $$e_{\bar{p}}\, e_p\, e_{\bar{p}} = \tau e_{\bar{p}}$$

(8.16) $$e_p\, e_{\bar{p}}\, e_p = \tau\,(1 - e_1 \vee \ldots \vee e_{p-2})\, e_p.$$

Conditions (8.13-8.16) together with the Temperley Lieb relations for $e_1, e_2, \ldots$ in the presence of a Markov trace serve to characterize $A(T_{p,2,r})$ $r = 2,3, \ldots, \infty$.

Theorem 8.1.

Let $p \geq 2$, $\tau > 0$, and $e_1, e_2, \ldots, e_{\bar{p}}$ a sequence of projections satisfying

(8.17) $\qquad e_n e_m = e_m e_n, \quad m,n = 1,2,\ldots, |m-n| \geq 2$

(8.18) $\qquad e_n e_{\bar{p}} = e_{\bar{p}} e_n, \; n \neq p$

(8.19) $\qquad e_n e_{n\pm1} e_n = \tau e_n$

(8.20) $\qquad e_{\bar{p}} e_p e_{\bar{p}} = \tau e_{\bar{p}}$

(8.21) $\qquad e_p e_{\bar{p}} e_p = \tau (1 - e_1 \vee \ldots \vee e_{p-2}) e_p$.

(8.22) $\qquad$ Let $A(\tau,p) = C^*(1, e_1, e_2, \ldots, e_{\bar{p}})$

If $p = 2$, suppose also that $e_1 e_{\bar{p}} = 0$.

Then $A(\tau,p)$ is non-trivial if and only if

$$\beta = \tau^{-1/2} \in \{ \| T_{p,2,r} \| : r \geq 1 \} \cup [\| T_{p,2,\infty} \|, \infty) \qquad (8.23a)$$

where $T_{p,2,1} = A_{p+1}$. Moreover there exists a surjective $*$-homomorphism

(8.23b) $\qquad \psi : A(T_{p,2,r}) \oplus \mathbb{C}(1 - e_1 \vee \ldots \vee e_{p+r-2} \vee e_{\bar{p}}) \to A(\tau,p)$,

when $\beta = \| T_{p,2,r} \|$, $r < \infty$.

(8.23c) $\qquad A(T_{p,2,\infty}) \to A(\tau,p)$ when $\beta \geq \| T_{p,2,r} \|$.

If $r < \infty$, i.e. $\beta = \| T_{p,2,r} \| < \| T_{p,2,\infty} \|$, then (8.23b) is automatically an isomorphism.

Suppose there exists a trace tr on $A(\tau,p)$ such that

(8.24) $\qquad tr(xe_n) = \tau \, tr \, x, \quad x \in A(\tau,p)_n$

where
$$A(\tau,p)_n = \begin{cases} C^*(1, e_1, e_2, \ldots, e_{n-1}) & n < p \\ C^*(1, e_1, e_2, \ldots, e_{n-1}, e_{\bar{p}}) & n \geq p \end{cases}$$

(8.26).

Then

(8.26a) $\qquad 1 = e_1 \vee \ldots \vee e_{p+r-2} \vee e_{\bar{p}}$

(8.26b) $$A(\tau,p) \simeq \begin{cases} A(T_{p,2,r}), & \beta = \| T_{p,2,r} \| \quad 1 \le r < \infty \\ A(T_{p,2,\infty}), & \beta \ge \| T_{p,2,\infty} \| \end{cases}$$

We will give a constructive proof of (8.26), obtaining expressions for matrix units in $A(\tau,p)_n$ under conditions (8.17)-(8.21). This yields a *-homomorphism from $A(T_{p,2,r})$ into $A(\tau,p)$ for appropriate $r \le \infty$, depending on τ. This will be a *-isomorphism under the assumption of a Markov trace on $A(\tau,p)$, (8.24).

To describe the matrix units in $A(T_{p,2,r})$, it is convenient to label paths in the Bratteli diagram $\hat{T}_{p,2,\infty}$ by certain sequences of half-integers as follows. In the first place, if $\alpha,\beta \in \hat{T}^{(0)}_{p,2,\infty}$, are on level m, respectively n, where $m \le n$, let Path (v,w) denote the paths of length $n - m$ from v to w in $\hat{T}_{p,2,\infty}$. For $\alpha = (v,m)$ labeled as in Figure 3, put $n = (m - d(v))/2$. Then if

(8.27) $I = \{0,1,2,3, \ldots,p - 2,\varepsilon,p - 1,p, \ldots\}$

where $\varepsilon = p - 2 + \frac{1}{2}$, define

(8.28) $I_\alpha = \{i = (i_1, \ldots,i_n) \in I^n : i_n \le d(v) - \delta_{v,\hat{p}}, i_{n-1} \le i_n+1,\ldots, i_1 \le i_2 + 1\}$.

Then we may identify the sets Path $(*,\alpha)$ and I_α as illustrated in Figure 15.

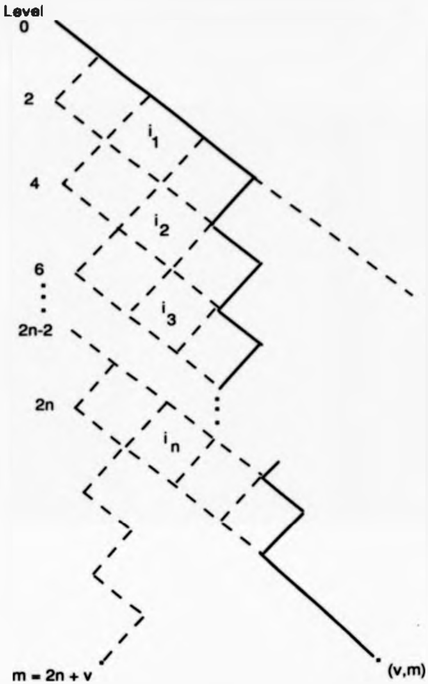

Figure 15

The numbers $i_1, i_2, \ldots, i_n$ correspond to the number of diamonds in the diagonal strip where

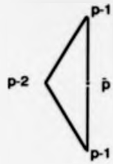

Figure 16

counts as half a diamond. For example when $p = 3$, $\alpha = (4,10)$, $i = (3/2,2,3/2)$ in I_α corresponds to the path in Figure 17:

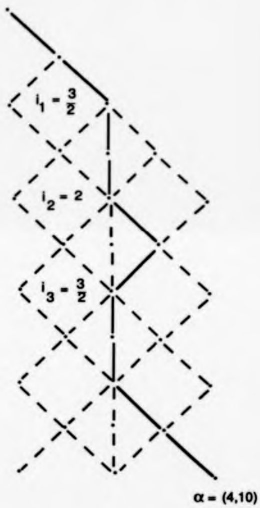

Figure 17

On $A(T_{p,2,r})$ we have an endomorphism obtained, essentially by shifting each vertex of a path down two levels, and then rejoining this path to $(*,0)$ via $(**,1)$. If $\alpha = (v,m) \in \hat{T}^{(0)}_{p,2,r}$, and $i \in \text{Path}\,(*,\alpha)$, i.e. $i = (i_1, i_2, \ldots, i_n)$, where $n = (m - d(v))/2$, then put $i' = (0, i_1, \ldots, i_n) \in \text{Path}\,(*,(v,m+2))$, as in Figure 18.

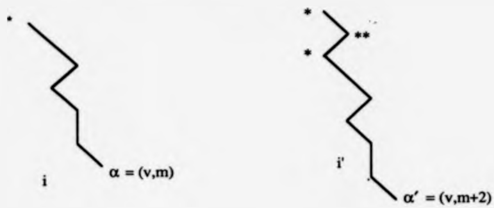

Figure 18

Then there exists an induced *-endomorphism of $A(T_{p,2,r})$ such that

$$\gamma(f_{i,j}) = f_{i',j'} \qquad (8.29).$$

One can obtain a formula, inductively, for the projection f_v corresponding to the extreme right hand path in terms of $1, e_1, e_2, \ldots, e_{\bar{p}}$. First take $f_0 = 1$, then suppose we have f_v for $1 \le v < p - 1$. On level $v + 1$ of $\hat{T}_{p,2,r}$, f_v splits into two paths, i.e. we have $f_v = f_{v+1} + i$, as shown in Figure 19.

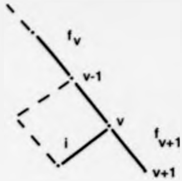

Figure 19

But the path i clearly corresponds to the projection $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ in End $t^2(t(v-1))$, and by (8.7) we see that, since $f_{v+1} e_v = 0$, we have

$$f_v e_v f_v = \frac{\phi_v}{\beta\phi_{v-1}} i \qquad (8.30).$$

It then follows that

$$f_{v+1} = f_v - \frac{\beta\phi_{v-1}}{\phi_v} f_v e_v f_v \qquad (8.31).$$

For $v = p - 1$, note that the path f_{p-1} splits as a sum of three paths on level p, as shown in Figure 20, i.e. $f_{p-1} = g_p + g_{\bar{p}} + i$

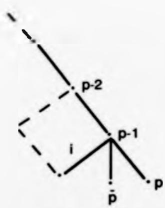

Figure 20

where $g_{\bar{p}} = e_{\bar{p}}$. Again it is clear that i corresponds to the projection $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ in End $(\ell^2(t(p-2)))$. Hence by (8.6)

$$i = \frac{\beta\phi_{p-2}}{\phi_{p-1}} f_{p-1}\, e_{p-1}\, f_{p-1},$$

and so

$$g_p = g_{p-1} - \frac{\beta\phi_{p-2}}{\phi_{p-1}} f_{p-1}\, e_{p-1}\, f_{p-1} - g_{\bar{p}} \qquad (8.32).$$

The situation for $v = p$ is similar to that for $v < p - 1$.

Consider, for $v \neq 0, p-1$, the operator $e_{v+1} g_{v+1}$ (where $g_v = f_v$, for $v = 0, \ldots, p-1$) contained in $A[v, v+2]$. This is given by

$$\frac{1}{\beta\phi_v} \begin{pmatrix} 0 & (\phi_{v-1}\phi_{v+1})^{\frac{1}{2}} \\ 0 & \phi_{v+1} \end{pmatrix} \qquad (8.33)$$

on End $(\ell^2(t(v)))$, and is zero on the other components in the decomposition (8.1). Defining

$$u_v = \frac{\beta\phi_v}{(\phi_{v-1}\phi_{v+1})^{\frac{1}{2}}} e_{v+1}\, g_{v+1} \qquad (8.34)$$

we see that $u_v^* \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, i.e. u_v^* flips the left hand path of End $(\ell^2(t(v)))$ to the right hand path as shown in Figure 21.

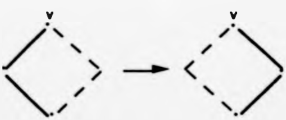

Figure 21

When $v = p - 1$, the operators $e_p\, g_{\bar{p}}$, $e_p\, g_p$ are both elements of $A[p - 1, p + 1]$, and are only non-zero on the component $\mathrm{End}\,(\ell^2(\iota(p - 1)))$. It is clear from (8.5) that, if we define

$$u_\varepsilon = \frac{\beta\phi_{p-1}}{\sqrt{(\phi_{p-2}\phi_{\bar{p}})}}\, e_p\, g_{\bar{p}} \tag{8.35}$$

$$u_{p-1} = \frac{\beta\phi_{p-1}}{\sqrt{(\phi_{p-2}\phi_{p})}}\, e_p\, g_p \tag{8.36}$$

where $\varepsilon = p - 2 + \frac{1}{2}$, then $u^*_\varepsilon \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$, and $u^*_{p-1} \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$. This is illustrated in Figure 22.

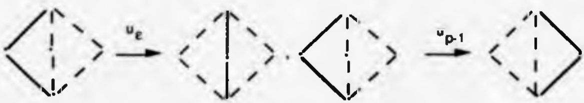

Figure 22

Note also, that for $v \neq \varepsilon$, $s_v = \left(\frac{\phi_{v-1}}{\beta\phi_v}\right) u_v$ is a partial isometry with final projection, $s_v s_v^* = e_{v+1} f_v$, and initial projection, $s_v^*\, s_v = \frac{\beta\phi_v}{\phi_{v+1}} f_{v+1}\, e_{v+1}\, f_{v+1}$. Also $s_\varepsilon = \left(\frac{\phi_{p-2}}{\beta\phi_{p-1}}\right)^{\frac{1}{2}} u_\varepsilon$ is a partial isometry with final projection, $s_\varepsilon s_\varepsilon^* = e_p g_{p-1}$, and initial projection, $s_\varepsilon^* s_\varepsilon = e_{\bar{p}}$.

Matrix units for $A(T_{p,2,r})$ are constructed as follows. Let $\alpha = (v,m) \in \hat{T}^{(0)}_{p,2,r}$, then if $n = (m - d(v))/2$, put $G_\alpha = \gamma^n(g_v)$, then by considering Figure 18, we see that G_α corresponds to the path shown in Figure 23. Put $\Delta_k = u_1 u_2 \ldots u_k$. To obtain an expression for G_α^i corresponding to the path $i = (i_1, i_2, \ldots, i_n)$ shown in Figure 15, one conjugates by the operator $\gamma^{n-1}(\Delta_{i_n})\gamma^{n-2}(\Delta_{i_{n-1}}) \ldots \gamma(\Delta_{i_2})\Delta_{i_1}$. Thus, $\gamma^{n-1}(\Delta_{i_n})^* G_k \gamma^{n-1}(\Delta_{i_n})$ corresponds to the path obtained from that in Figure 23 by flipping i_n diamonds in the n^{th} diagonal strip shown in Figure 15. Conjugating the new path by $\gamma^{n-2}(\Delta_{i_{n-1}})$, flips i_{n-1} diamonds in the $(n - 1)^{th}$ strip etc.

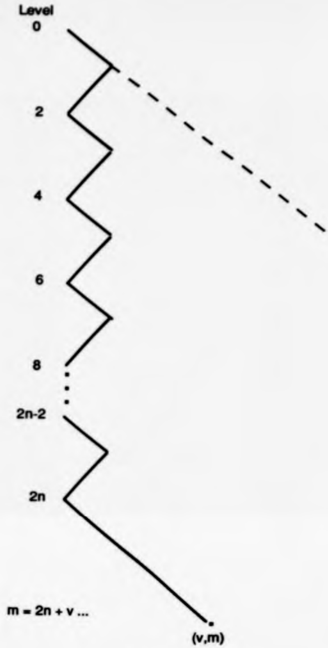

Figure 23

Proof of Theorem 8.1:

Lemma 8.2.

Let $p = 3,4, \ldots$, then $e_m e_{\bar p} = 0, \quad m = 1,2, \ldots, p-1.$

Proof:

We see from (8.21) that

$$e_m e_p e_{\bar p} e_p = 0, \quad m = 1,2, \ldots, p-2.$$

Thus

$$(e_m e_p e_{\bar p})(e_{\bar p} e_p e_m) = 0$$

shows that $e_m e_p e_{\bar p} = 0$. Then using $[e_m, e_{\bar p}] = 0$, and $e_{\bar p} e_p e_{\bar p} = \tau e_{\bar p}$ we see that $e_m e_{\bar p} = 0$ for $m = 1,2, \ldots, p-2$. In particular $e_{p-2} e_{\bar p} = 0$. Consequently

$$\tau e_{p-1} e_{\bar p} = e_{p-1} e_{p-2} e_{p-1} e_{\bar p} = 0$$

as $[e_{p-1}, e_{\bar p}] = 0$.

Lemma 8.3.

Let $p \geq 2$, $\tau > 0$, such that $e_{\bar p}, e_1, e_2, \ldots$ is a sequence of projections satisfying (8.17) - (8.21). Define

(8.37) $\qquad \gamma_n(x) = \tau^{-(n-1)} e_1 e_2 \cdots e_n x e_n \cdots e_2 e_1, \quad x \in A(\tau,p).$

Then there exists an unique *-endomorphism γ of $A(\tau,p)$ such that

(8.38) $\qquad \gamma(x) = \lim_{n \to \infty} \gamma_n(x), \quad x \in A(\tau,p)$

(8.39) $\qquad \gamma(x) = \gamma_n(x), \quad x \in A(\tau,p)_{n-1}$

(8.40) $\qquad \gamma(1) = e_1$

(8.41) $\qquad \gamma(e_m) = e_1 e_{m+2}$

(8.42) $\qquad \gamma(e_{\bar p}) = \tau^{-p} e_1 e_2 \cdots e_{p-1} e_p e_{\bar p} e_{p+1} e_p e_{p-1} \cdots e_2 e_1.$

Proof:

Let A_o denote the set of $y \in A(\tau,p)$ such that $\lim_{n\to\infty} \gamma_n(y)$ exists. For $y \in A_o$, let $\gamma(y) = \lim_{n\to\infty} \gamma_n(y)$. Then elementary computations show that $1, e_{\bar{p}}, e_1, e_2, \ldots \in A_o$, (8.40)-(8.42) hold, and indeed

(8.43) $\qquad \gamma_n(1) = e_1, \qquad n \geq 1$

(8.44) $\qquad \gamma_n(e_m) = e_1\, e_m \quad n > m+1$

(8.45) $\qquad \gamma_n(e_{\bar{p}}) = \tau^{-p}\, e_1\, e_2 \ldots e_{p-1}\, e_p\, e_{p+1} e_{\bar{p}}\, e_p\, e_{p-1} \ldots e_2\, e_1, \; n > p.$

Then if $x,y \in A_o$,

(8.46) $\qquad e_n \ldots e_2\, e_1\, e_2 \ldots e_n = \tau^n\, e_n,$

c.f. (8.43), and so:

$$\gamma_n(x)\, \gamma_n(y) = \tau^{-n}\, e_1\, e_2 \ldots e_n\, x\, e_n\, y\, e_n \ldots e_1 \tag{8.47}$$

But $[e_n,x] \to 0$ as $n \to \infty$, for any $x \in A(\tau,p)$, as $[e_n,e_v] = 0$ for n large, $v \in T^{(0)}_{p,2,\infty}$. Thus $xy \in A_o$ and $\gamma(xy) = \gamma(x)\,\gamma(y)$. Thus A_o is a dense *-subalgebra of $A(\tau,p)$ and (8.39) holds. Now

$$\begin{aligned} \|\tau^{-n}\, e_1 \ldots e_n\, x\, e_n \ldots e_1\| &\leq \tau^{-n}\, \|e_1 \ldots e_n\|^2\, \|x\| \\ &= \tau^{-n}\, \|e_1 \ldots e_n\, e_n \ldots e_1\|\, \|x\| \\ &= \|e_1\|\, \|x\| \quad \text{by (8.46).} \end{aligned}$$

Hence γ_n is a contraction, and so A_o is closed. Thus $A_o = A(\tau,p)$ and the Lemma follows.

Suppose $e_{\bar{p}}, e_1, e_2, \ldots,$ is a sequence of projections satisfying (8.17)-(8.21) where $\tau^{-\frac{1}{2}} = \beta$ is such that $\varphi_v(\beta) \neq 0$ for all $v \in T^{(0)}_{p,2,r-1}$, and some $r \geq 2$, where $\{\varphi_v(x) : v \in T^{(0)}_{p,2,\infty}\}$ is the family of rational functions associated with the graph $T_{p,2,\infty}$ as in Lemma 3.1. Then we can define a sequence of operators $g_v \in A(\tau,p)_{d(v)}$ for $v \in T^{(0)}_{p,2,r}$ by

(8.48) $\quad g_v = f_v \qquad\qquad v = 0,1, \ldots, p-1$

(8.49) $\quad g_{\bar{p}} = e_{\bar{p}}$

(8.50) $$g_p = g_{p-1} - \beta \frac{\phi_{p-2}}{\phi_{p-1}} g_{p-1} e_{p-1} g_{p-1} - g_{\bar{p}}$$

(8.51) $$g_{v+1} = g_v - \beta \frac{\phi_{v-1}}{\phi_v} g_v e_v g_v \qquad\qquad p \le v \le p+r-3$$

where $\varphi_j = \varphi_j(\beta)$.

Lemma 8.4.

Under the preceding conditions, the family $\{e_k, g_v,\ k,v \in T^{(0)}_{p,2,r-1},\ k \neq \bar{p}\}$ satisfy

(a) $\quad e_k\, g_v = g_v\, e_k \qquad v = 0,1, \ldots, p+r-2, k \ge v+1$

(b) (i) $\quad e_p\, g_{\bar{p}}\, e_p = (\phi_{\bar{p}} / \beta\phi_{p-1})\, e_p\, g_{p-1}$

(ii) $\quad e_v\, g_v\, e_v = (\phi_v / \beta\phi_{v-1})\, e_v\, g_{v-1} \qquad v = 1,2, \ldots, p+r-2$

(c) $\quad e_k\, g_v = 0 \qquad v = 2,3, \ldots, p+r-2,\ k = 1,2, \ldots, v-1$

(d) $\quad g_v^2 = g_v = g_v^* \qquad v \in T^{(0)}_{p,2,r-1} = \{0,1, \ldots, p+r-2, \bar{p}\}$

(e) (i) $\quad g_{\bar{p}}\, g_v = g_{\bar{p}} \qquad v = 0,1, \ldots, p-1$

(ii) $\quad g_{\bar{p}}\, g_v = 0 \qquad v = p, p+1, \ldots, p+r-2$

(iii) $\quad g_k\, g_v = g_v \qquad v = 0,1, \ldots, p+r-2,\ k = 1,2, \ldots, v.$

(f) $$g_v = \begin{cases} 1 - e_1 \vee \ldots \vee e_{v-1} & v = 2,3, \ldots, p-1 \\ 1 - e_1 \vee \ldots \vee e_{v-1} \vee e_{\bar{p}} & v = p, p+1, \ldots . \end{cases}$$

Proof:

For $v = 0,1,2, \ldots, p-1$, the relevant parts of the lemma are clear. Next note that $\phi_{\bar{p}} = \beta^{-1}\phi_{p-1}$, and so (b)(i) follows immediately from (8.21). To see (e)(i), note that $g_v = 1 - e_1 v \ldots v e_{v-1}$, for $v = 2,3, \ldots, p-1$, and so $g_v g_{\bar{p}} = (1 - e_1 v \ldots v e_{v-1}) e_{\bar{p}} = e_{\bar{p}} = g_{\bar{p}}$ by (8.5). Moreover, to show (e)(ii):

$$\begin{aligned} g_{\bar{p}} g_p &= g_{\bar{p}}(g_{p-1} - (\beta\varphi_{p-2}/\varphi_{p-1})\, g_{p-1}\, e_{p-1}\, g_{p-1} - g_{\bar{p}}) \\ &= g_{\bar{p}} - (\beta\varphi_{p-2}/\varphi_{p-1})\, g_{\bar{p}} e_{p-1}\, g_{p-1} - g_{\bar{p}} = 0 \end{aligned}$$

since $g_{\bar{p}}\, g_{p-1} = g_{\bar{p}}$, and $g_{\bar{p}}\, e_{p-1} = e_{\bar{p}}\, e_{p-1} = 0$ by Lemma 8.2. It follows inductively on $v = p, p+1, \ldots, p+r-3$ using (8.15) that $g_{\bar{p}}\, g_v = 0$ for such v, i.e. (e)(ii) holds.

We now prove the properties listed for g_p. It is clear from (8.48-50) that g_p is in the algebra generated by $1, e_1, e_2, \ldots, e_{p-1}$ and $e_{\bar{p}}$. Thus (a) holds for $v = p$. Next, since e_p and g_{p-1} commute, we have

$$\begin{aligned} e_p g_p e_p &= e_p[g_{p-1} - (\beta\varphi_{p-2}/\varphi_{p-1})\, g_{p-1}\, e_{p-1}\, g_{p-1} - g_{\bar{p}}]\, e_p \\ &= e_p\, g_{p-1} - (\beta\varphi_{p-2}/\varphi_{p-1})\, g_{p-1} e_p\, e_{p-1} e_p\, g_{p-1} - e_p\, g_{\bar{p}} e_p \\ &= e_p g_{p-1} - (\varphi_{p-2}/\beta\varphi_{p-1}) g_{p-1} e_p\, g_{p-1} - (\varphi_{\bar{p}}/\beta\varphi_{p-1})\, e_p g_{p-1} \quad \text{using (b)(i)} \\ &= [1 - (\varphi_{p-2}/\beta\varphi_{p-1}) - (\varphi_{\bar{p}}/\beta\varphi_{p-1})\, e_p g_{p-1}. \end{aligned}$$

But $\varphi_p = \beta\varphi_{p-1} - \varphi_{p-2} - \varphi_{\bar{p}}$, and so we obtain (b)(ii) for $v = p$.

We know that (c) holds for $v \le p-1$ by definition of g_i (8.12), and $e_k\, g_{\bar{p}} = 0$ by Lemma 8.2 for $k = 1,2, \ldots, p-1$. Thus $e_k g_p = 0$ by Lemma 8.4(c), for $k = 1,2, \ldots, p-2$. Also we have, using (b)(ii) for $v = p-1$, and noting that $e_{p-1} g_{\bar{p}} = 0$ that

$$\begin{aligned} e_{p-1} g_p &= e_{p-1} g_{p-1} - (\beta\varphi_{p-2}/\varphi_{p-1})\, e_{p-1}\, g_{p-1}\, e_{p-1} g_{p-1} - e_{p-1} g_{\bar{p}} \\ &= e_{p-1} g_{p-1} - e_{p-1}\, g_{p-2}\, g_{p-1}. \end{aligned}$$

But $g_{p-2}\, g_{p-1} = g_{p-1}$ by (e)(iii) for $v = p-1$, $k = p-2$, and so $e_{p-1} g_p = 0$. Thus (c) holds for $v = p$.

Next note that by (b)(ii) for $v = p - 1$, one easily shows that $g_{p-1} - (\beta\varphi_{p-2}/\varphi_{p-1})\, g_{p-1}\, e_{p-1}\, g_{p-1}$ is a projection. Consequently,

$$g_p^2 = g_{p-1} - (\beta\varphi_{p-2}/\varphi_{p-1})\, g_{p-1}\, e_{p-1}\, g_{p-1} - 2g_{\bar p}[g_{p-1} - (\beta\varphi_{p-2}/\varphi_{p-1})\, g_{p-1}\, e_{p-1}\, g_{p-1}] + g_{\bar p}$$

$$= g_{p-1} - (\beta\varphi_{p-2}/\varphi_{p-1})\, g_{p-1}\, e_{p-1}\, g_{p-1} - 2g_{\bar p} + g_{\bar p} = g_p.$$

Here we have used Lemma 8.2 and the fact that $g_{p-1}\, g_{\bar p} = g_{\bar p}$ by (e)(i). This gives (d) for $v = p$, and (e) for $v = p$ is clear.

Now suppose, for some v, $p < v < p + r - 2$, that g_v has the properties listed. Then we show that g_{v+1} also satisfies these properties. In the first place, (a) follows from the definition of g_{v+1}. Then since $\varphi_{v+1} = \beta\varphi_v - \varphi_{v-1}$, for $v > p$, we have

$$\begin{aligned} e_{v+1}\, g_{v+1}\, e_{v+1} &= e_{v+1}[g_v - (\beta\varphi_{v-1}/\varphi_v)\, g_v\, e_v\, g_v]\, e_{v+1} \\ &= g_v\, e_{v+1} - (\beta\varphi_{v-1}/\varphi_v)\, g_v\, e_{v+1}\, e_v\, e_{v+1}\, g_v \\ &= g_v\, e_{v+1} - (\beta\varphi_{v-1}/\varphi_v)\, \beta^{-2}\, g_v\, e_{v+1}\, g_v \\ &= (1 - \varphi_{v-1}/\beta\varphi_v)\, g_v\, e_{v+1} \\ &= (\varphi_{v+1}/\beta\varphi_v)\, g_v\, e_{v+1}. \end{aligned}$$

Next, by the inductive hypothesis, we have $e_j\, g_v = 0$ for $1 \le j \le v - 1$, and so $e_j\, g_{v+1} = 0$ for $1 \le j \le v - 1$. Moreover, by b(ii), and e(iii) for v,

$e_v\, g_{v+1} = e_v\, g_v - (\beta\varphi_{v-1}/\varphi_v)\, e_v\, g_v e_v\, g_v = e_v\, g_v - e_v\, g_{v-1} g_v = e_v\, g_v - e_v\, g_v = 0.$

Thus (c) holds for $v + 1$. For (d), one has, using $g_v^2 = g_v$ and (b)(ii) for v that

$$\begin{aligned} g_{v+1}^2 &= g_v - (2\beta\varphi_{v-1}/\varphi_v)\, g_v e_v\, g_v + (\beta\varphi_{v-1}/\varphi_v)^2\, g_v e_v\, g_v e_v\, g_v \\ &= g_v - (2\beta\varphi_{v-1}/\varphi_v)\, g_v e_v\, g_v + (\beta\varphi_{v-1}/\varphi_v)\, g_v e_v\, g_{v-1} g_v = g_{v+1}. \end{aligned}$$

Finally (e) for $v + 1$ is clear.

It follows from (c) and e(ii) that $1 - g_v$ is an upper bound for $e_1, e_2, \ldots, e_{v-1}, e_{\bar{p}}$. To show that it is the least upper bound note that $1 - g_v$ is a linear combination of monomials in $e_1, e_2, \ldots, e_{v-1}, e_{\bar{p}}$.

Let $p, r \geq 2$ be fixed, $\beta > 0$ with $\varphi_v(\beta) > 0$ for $v \in T^{(0)}_{p,2,r}$. Put $\varepsilon = p - 2 + \frac{1}{2}$. Then we define operators $u_1, u_2, \ldots, u_{p+r-3}, u_\varepsilon, \bar{u}_\varepsilon$ as follows:

(8.53) $$u_k = \beta\varphi_k(\varphi_{k-1}\varphi_{k+1})^{-\frac{1}{2}} e_{k+1}\, g_{k+1}, \qquad k = 1,2, \ldots, p + r - 3$$

(8.54) $$u_\varepsilon = \beta\varphi_{p-1}\,(\varphi_{p-2}\varphi_{\bar{p}})^{-\frac{1}{2}} e_p\, g_{\bar{p}}$$

(8.55) $$\bar{u}_\varepsilon = \beta\varphi_{p-1}\,(\varphi_{\bar{p}}\varphi_p)^{-\frac{1}{2}} e_p\, g_p.$$

Note that $u_k \in A(\tau,p)_{k+2}$, $k = 1,2, \ldots$, $u_\varepsilon \in A(\tau,p)_\varepsilon$, and

(8.56) $$u_\varepsilon \bar{u}_\varepsilon = u_{p-1}$$

from Lemma 8.4 (b)(i) and (e)(iii). For $k = 1,2, \ldots, p + r - 3$ put

(8.57) $$\Delta_k = u_1 u_2 \cdots u_k.$$

(8.58) $$\Delta_\varepsilon = u_1 u_2 \cdots u_{p-2}\, u_\varepsilon.$$

Lemma 8.5.

Let $p,r \geq 2$ be fixed, and $e_1, e_2, \ldots, e_{\bar{p}}$, a sequence of projections satisfying (8.17)-(8.21) where $\tau = \beta^{-2} > 0$, and $\varphi_v(\beta) > 0$ for all $v \in T^{(0)}_{p,2,r}$. Then we have:

(a) $$\Delta_k \Delta_v^* = 0 \text{ for } k,v \in \{1,2, \ldots, p + r - 3, \varepsilon\}, \text{ and } k \neq v.$$

(b) $$\Delta_k\, g_v\, \Delta_k^* = \Delta_k \Delta_k^*, \quad v \neq \bar{p},\ 1 \leq v \leq k + 1$$

$$\Delta_\varepsilon\, g_{\bar{p}}\, \Delta_\varepsilon^* = \Delta_\varepsilon \Delta_\varepsilon^*, \quad k = \varepsilon,$$

(c) $\qquad e_1 \Delta_k \overset{*}{\Delta_k} e_1 = \gamma(g_k), \quad k = 1, \ldots, p + r - 3$

$\qquad e_1 \Delta_\varepsilon \overset{*}{\Delta_\varepsilon} e_1 = \gamma(g_{p-1})$

(d) $\qquad u_k \gamma(x) = 0, \quad x \in A(\tau, p), \; k \geq 1.$

(e) $\qquad u_i g_v = 0, \quad v \in T^{(0)}_{p,2,r} \; d(v) > i + 1, i \neq \varepsilon.$

$\qquad u_\varepsilon g_v = 0, \quad v = p, p + 1, \ldots$

(f) $\qquad \overset{*}{\Delta_v} \gamma(g_v) \Delta_v = (\beta\varphi_v/\varphi_{v+1}) f_{v+1} e_{v+1} f_{v+1}, \; v \in T^{(0)}_{p,2,r}.$

$\qquad v \neq 0, \bar{p}, \; p + r - 2$ (if $r < \infty$).

(g) $\qquad \overset{*}{\Delta_\varepsilon} \gamma(g_{p-1}) \Delta_\varepsilon = g_{\bar{p}}.$

Proof:

(a) For $v \neq \varepsilon$,

$$u_\varepsilon \overset{*}{u_v} = \beta\varphi_{p-1}(\varphi_{p-2}\,\varphi_{\bar{p}})^{-\frac{1}{2}}\beta\varphi_v(\varphi_{v-1}\varphi_{v+1})^{-\frac{1}{2}} e_p\, g_{\bar{p}}\, g_{v+1}\, e_{v+1} = 0$$

since if $v \leq p - 2$, $g_{\bar{p}} g_{v+1} = g_{\bar{p}}$ and $g_{\bar{p}} e_{v+1} = 0$, whereas if $v \geq p - 1$, we have $g_{\bar{p}} g_{v+1} = 0$. Similarly for $v, k \neq \varepsilon$, we have, assuming that $k < v$:

$$u_k \overset{*}{u_v} = \beta\varphi_k(\varphi_{k-1}\,\varphi_{k+1})^{-\frac{1}{2}}\beta\varphi_v(\varphi_{v-1}\varphi_{v+1})^{-\frac{1}{2}} e_{k+1}\, g_{k+1}\, g_{v+1}\, e_{v+1} = 0$$

since $g_{k+1} g_{v+1} = g_{v+1}$, and $e_{k+1} g_{v+1} = 0$.

(b) For $v \neq \bar{p}$, one has $u_k g_v \overset{*}{u_k} = u_k \overset{*}{u_k}$ for $k \leq v + 1$, since $g_k g_{v+1} = g_{v+1}$ for $1 \leq v \leq k + 1$. For $1 \leq k \leq p - 1$, one has $u_k g_{\bar{p}} \overset{*}{u_k} = 0$, since $g_{k+1} g_{\bar{p}} = g_{\bar{p}}$, and $g_{\bar{p}} e_{k+1} = 0$ for $1 \leq k \leq p - 2$, and $g_{k+1} g_{\bar{p}} = 0$ for $k = p - 1$. Moreover $u_\varepsilon g_{\bar{p}} \overset{*}{u_\varepsilon} = u_\varepsilon \overset{*}{u_\varepsilon}$ as $g_{\bar{p}}^2 = g_{\bar{p}}$.

(c) For $k = 1, 2, \ldots, p + r - 3$:

$$\Delta_k \overset{*}{\Delta_k} = u_1 u_2 \cdots u_k \overset{*}{u_k} \cdots \overset{*}{u_2} \overset{*}{u_1}$$

$$= \eta_k e_2 g_2 e_3 g_3 \cdots e_{k+1} g_{k+1} e_{k+1} \cdots g_3 e_3 g_2 e_2$$

$$= \eta_k e_2 e_3 \cdots e_{k+1} g_2 g_3 \cdots g_k g_{k+1} g_k \cdots g_3 g_2 e_{k+1} \; e_3 e_2$$

$$= \eta_k e_2 e_3 \cdots e_{k+1} g_{k+1} e_{k+1} \cdots e_2 \tag{8.59}$$

where we have used Lemma 8.1(a) and (e), and

$$\eta_k = \frac{(\beta\varphi_1)^2}{\varphi_o\,\varphi_2} \;\cdots\cdots\; \frac{(\beta\varphi_k)^2}{\varphi_{k-1}\,\varphi_{k+1}} \tag{8.60}.$$

Then from (8.23) and Lemma 8.1(b)(ii) we obtain

$$\Delta_k \Delta_k^* = \eta_k(\varphi_{k+1}/\beta\varphi_k)\, e_2 e_3 \cdots e_k e_{k+1} g_k e_k \cdots e_2$$

and so by Lemma 8.2 we have

$$e_1 \; \Delta_k \Delta_k^* e_1 = \eta_k(\varphi_k/\beta\varphi_{k-1})\, e_1 e_2 e_3 \cdots e_k e_{k+1} f_k e_{k+1} e_k \cdots e_2$$

$$= \eta_k(\varphi_{k+1}/\beta\varphi_k)\, \beta^{-2k}\, \gamma(f_k).$$

But

$$\eta_k \cdot \frac{\varphi_k}{\beta\varphi_{k-1}} \cdot \frac{1}{\beta^{2k}} = \beta^{2k} \cdot \frac{\varphi_1^2\varphi_2^2 \cdots \varphi_k^2}{\varphi_0\varphi_2\varphi_1\varphi_3 \cdots \varphi_{k-1}\varphi_{k+1}} \cdot \frac{\varphi_{k+1}}{\beta\varphi_k} \cdot \frac{1}{\beta^{2k}} = 1 \tag{8.61}$$

which establishes (c) for $k \neq \varepsilon$.

Similarly, one uses $e_p \, g_{\bar{p}} \, e_p = (\varphi_{\bar{p}}/\beta\varphi_{p-1})\, e_p\, g_{p-1}$ and Lemma 8.2 to show that $e_1 \, \Delta_\varepsilon \Delta_\varepsilon^* \, e_1 = \gamma(g_{p-1})$.

(d) This follows because $\gamma(x) = e_1\gamma(x)$ by (8.16), and $g_{k+1}\, e_1 = 0 = g_{\bar{p}}\, e_1$ for $k = 1,2, \ldots, p + r - 3$.

(e) This follows immediately from Lemma 8.4(c), and (e).

(f) We have by Lemma 8.2, and Lemma 8.1(a) and (e) that for v as stated:

$$\Delta_v^* \gamma(g_v) \Delta_v = \beta^{2v} \Delta_v^* e_1 e_2 \cdots e_{v+1} g_v e_{v+1} \cdots e_2 e_1 \Delta_v$$

$$= \beta^{2v} \eta_v \, g_{v+1} e_{v+1} \cdots g_2 e_2 e_1 e_2 \cdots e_{v+1} g_v e_{v+1} \cdots e_2 e_1 e_2 g_2 \cdots e_{v+1} \, g_{v+1}$$

$= \beta^{2v}\eta_v\, g_{v+1}g_v \cdots g_2 e_{v+1} \cdots e_2 e_1 e_2 \cdots e_{v+1} g_v e_{v+1} \cdots e_2 e_1 e_2 \cdots e_{v+1}\, g_2 \cdots g_{v+1}$

$= \beta^{2v}\eta_v\, g_{v+1}e_{v+1}\cdots e_2e_1e_2\cdots e_{v+1}g_ve_{v+1}\cdots e_2e_1e_2\cdots e_{v+1}\, g_2\cdots g_{v+1}$

where $\eta_v = \beta^{2v}(\beta\varphi_v/\varphi_{v+1})$ by (8.61). But using (8.46) and Lemma 8.4(a) and (e)(iii) we have

$$\begin{aligned}\Delta_v^{*}\gamma(g_v)\Delta_v &= \beta^{-2v}\eta_v\, g_{v+1}\, e_{v+1}\, g_v\, e_{v+1}\, g_{v+1} \\ &= (\beta\varphi_v/\varphi_{v+1})\; g_{v+1}\, g_v\, e_{v+1}\, g_{v+1} \\ &= (\beta\varphi_v/\varphi_{v+1})\; g_{v+1}\, e_{v+1}\, g_{v+1}.\end{aligned}$$

(g) Similarly, we have

$$\Delta_\varepsilon\,\gamma(g_{p-1})\,\Delta_\varepsilon^{*} = \beta^{2(p-1)}\Delta_\varepsilon e_1e_2\cdots e_pg_{p-1}e_p\cdots e_2e_1\Delta_\varepsilon^{*}$$

$= \beta^{2(p-1)}\beta^{2(p-1)}(\phi_1\phi_{p-1}/\phi_o\phi_{\bar p})\, g_{\bar p}e_pg_{p-1}e_{p-1}\cdots g_2e_2e_1e_2\cdots e_pg_{p-1}e_p\cdots e_1e_2g_2 \cdots e_pg_{\bar p}$

$= \beta^{2(p-1)}\beta^{2p-1}(\phi_{p-1}/\phi_{\bar p})\, g_{\bar p}g_{p-1}\cdots g_2e_pe_{p-1}\cdots e_2e_1e_2\cdots e_pg_{p-1}e_p\cdots e_2e_1e_2\cdots e_pg_2\cdots g_{p-1}g_{\bar p}$

$= \beta^{-2(p-1)}\,\beta^{2p-1}\,(\phi_{p-1}/\phi_{\bar p})\; g_{\bar p}\,e_p\,g_{p-1}\,e_p\,g_{\bar p}$

$= \beta^2\,\dfrac{\phi_{p-1}}{\beta\phi_{\bar p}}\; g_{\bar p}\,e_p\,g_{p-1}\,e_p\,g_{\bar p}\,.$

But $\phi_{p-1} = \beta\phi_{\bar p}$, and by Lemma 8.4 we have

$$g_{\bar p}\,e_p\,g_{p-1}\,e_p\,g_{\bar p} = g_{\bar p}\,e_p\,g_{\bar p} = \beta^{-2}\,g_{\bar p}$$

and the result follows.

Let p,r, and $\tau = \beta^{-2}$ be fixed, where $p \ge 2$, $2 \le r \le \infty$, and suppose that

(8.62) $$\phi_v(\beta) > 0 \text{ for all } v \in T^{(0)}_{p,2,r}\,.$$

Recall from Section 3, that if $r < \infty$, and $\beta = ||T_{p,2,r}||$, or if $r = \infty$, and $\beta \geq ||T_{p,2,\infty}||$, then (8.62) is true.

Let $\alpha = (v,m) \in \hat{T}^{(0)}_{p,2,r}$, and put $n = (m - d(v))/2$. Then define

$$G_\alpha = \gamma^n (g_v) \tag{8.63}$$

and for $i,j \in I_\alpha = \text{Path}\,(*,\alpha)$ define

$$G^{ij}_\alpha = \Delta^*_{i_1}\, \gamma(\Delta_{i_2})^* \cdots \gamma^{n-1}(\Delta_{i_n})^* \gamma^n (g_v)\, \gamma^{n-1}(\Delta_{j_n}) \cdots \gamma(\Delta_{j_2})\, \Delta_{j_1}. \tag{8.64}$$

Note that $G^{oo}_\alpha = G_\alpha$, and $(G^{ij}_\alpha)^* = G^{ji}_\alpha$. Put $G^i_\alpha = G^{ii}_\alpha$,

and $v_{\alpha,i} = G_\alpha\, \gamma^{n-1}(\Delta_{i_n}) \cdots \gamma(\Delta_{i_2})\, \Delta_{i_1}$.

Lemma 8.6.

For $\alpha = (v,m)$, $\beta = (w,m) \in \hat{T}^{(0)}_{\beta,2,r}$, $i, j \in \text{Path}\,(*,\alpha)$, $k, \ell \in \text{Path}\,(*,\beta)$ we have

$$G^{ij}_\alpha G^{k\ell}_\beta = \delta_{\alpha\beta}\, \delta_{jk}\, G^{i\ell}_\alpha . \tag{8.65}$$

Proof:

(a) $G^{ij}_\alpha\, G^{j\ell}_\alpha = G^{i\ell}_\alpha$.

It is clear that $G^2_\alpha = G_\alpha$. Now suppose that $n \geq 1$, and $i \neq 0$. We prove by induction on n that

$$e_1 e_3 \cdots e_{2n-1}\, \gamma^{n-1}(\Delta_{i_n}) \cdots \Delta_{i_1} \Delta^*_{i_1} \cdots \gamma^{n-1}(\Delta_{i_n})^*\, e_{2n-1} \cdots e_3 e_1 = \gamma^n (g_z) \tag{8.66}$$

where $z = [i_n]$. If $n = 1$, this follows immediately from Lemma 8.5(c) since $i_1 \neq 0$. Now suppose that (8.66) is true for $n > 1$. Then by the induction hypothesis, and Lemma 8.5(c) we have

$$e_1 e_3 \cdots e_{2n+1}\, \gamma^n (\Delta_{i_{n+1}}) \cdots \Delta_{i_1} \Delta^*_{i_1} \cdots \gamma^n (\Delta_{i_{n+1}})^*\, e_{2n+1} \cdots e_3 e_1$$

$$= e_{2n+1}\gamma^n(\Delta_{i_{n+1}})[e_1e_3 \cdots e_{2n-1}\gamma^{n-1}(\Delta_{i_n}) \cdots \Delta_{i_1}\Delta_{i_1}^* \cdots \gamma^{n-1}(\Delta_{i_n})^* e_{2n-1} \cdots e_3e_1]$$
$$\gamma^n(\Delta_{i_{n+1}})^* e_{2n+1}$$

$$= e_{2n+1}\,\gamma^n(\Delta_{i_{n+1}})\,\gamma^n(g_z)\,\gamma^n(\Delta_{i_{n+1}}^*)\,e_{2n+1}$$

$$= \gamma^n(e_1\,\Delta_{i_{n+1}}\,g_z\,\Delta_{i_{n+1}}^*e_1).$$

But $z \le i_{n+1} + 1$, and so by Lemma 8.5(b) and (c) we have

$$e_1\,\Delta_{i_{n+1}}\,g_z\,\Delta_{i_{n+1}}^*\,e_1 = e_1\,\Delta_{i_{n+1}}\,\Delta_{i_{n+1}}^*\,e_1 = \gamma(g_{z'}),$$

where $z' = [i_{n+1}]$. Hence (8.66) is true for all n. Now it follows from (8.66), noting that $G_\alpha = \gamma^n(g_v) = e_1\,e_3 \cdots e_{2n-1}\,\gamma^n(g_v)$, that

$$v_{\alpha,i}\,v_{\alpha,i}^* = G_\alpha\,\gamma^n(g_z)\,G_\alpha = \gamma^n(g_v\,g_z\,g_v) = \gamma^n(g_v) = G_\alpha$$

since $z \le d(v)$. This gives (a).

(b) $G_\alpha^{ij}\;G_\alpha^{k\ell} = 0$, for $j \ne k$.

We may assume that $n \ge 1$. We show that for $i \ne j$, $v_{\alpha,i}\,v_{\alpha,j}^* = 0$. If $i \ne j$, then there exists a $k \le n$ such that $i_1 = j_1, \ldots, i_k = j_k$, and $i_{k+1} \ne j_{k+1}$. Then by (8.66) we have

$$v_{\alpha,i}\,v_{\alpha,j}^* = G_\alpha\,\gamma^{n-1}(\Delta_{i_n}) \cdots \gamma^k(\Delta_{i_{k+1}})\,\gamma^k(g_z)\,\gamma^k(\Delta_{j_{k+1}})^* \cdots \gamma^{n-1}(\Delta_{j_n})^*\,G_\alpha.$$

But $z \le i_{k+1} + 1$, and so $\Delta_{i_{k+1}}\,g_z = \Delta_{i_{k+1}}$. It follows from Lemma 8.5(a) that

$$\gamma^k(\Delta_{i_{k+1}})\,\gamma^k(g_z)\,\gamma^k(\Delta_{j_{k+1}})^* = \gamma^k(\Delta_{i_{k+1}}\,g_z\Delta_{j_{k+1}}^*) = \gamma^k(\Delta_{i_{k+1}}\,\Delta_{j_{k+1}}^*) = 0$$

since $i_{k+1} \ne j_{k+1}$.

(c) $G_\alpha^{ij}\,G_\beta^{i'j'} = 0$, for $v \ne w$.

Put $k = (m - d(w))/2$. Suppose that $n = k$, then $v = p$, $w = \bar{p}$, or $v = \bar{p}$, and $w = p$. We must show that

$$v_{\alpha,i}\,v_{\beta,j}^* = G_\alpha\,\gamma^{n-1}(\Delta_{i_n}) \cdots \Delta_{i_1}\Delta_{j_1}^* \cdots \gamma^{n-1}(\Delta_{j_n})^*\,G_\beta$$

vanishes. First suppose that there exists $t < n$ such that $i_1 = j_1, \ldots, i_t = j_t$, and $i_{t+1} \neq j_{t+1}$. Then as in the proof of (a) we have

$$v_{\alpha,i} v^*_{\beta,j} = G_\alpha \gamma^{n-1}(\Delta_{i_n}) \ldots \gamma^t(\Delta_{i_{t+1}}) \gamma^t(g_{i_t}) \gamma^t(\Delta_{j_{t+1}})^* \ldots \gamma^{n-1}(\Delta_{j_n})^* G_\beta$$

if $i_t \neq e$, otherwise we replace $\gamma^t(g_{i_t})$ by $\gamma^t(g_{p-1})$. Then since $i_t \leq i_{t+1} + 1$, we see that $\Delta_{i_{t+1}} g_{i_t} = \Delta_{i_{t+1}}$, and so

$$v_{\alpha,i} v^*_{\beta,j} = G_\alpha \gamma^n(\Delta_{i_n}) \ldots \gamma^t(\Delta_{i_{t+1}} \Delta^*_{j_{t+1}}) \ldots \gamma^n(\Delta_{j_n})^* G_\beta.$$

But $i_{t+1} \neq j_{t+1}$, and so $\Delta_{i_{t+1}} \Delta^*_{j_{t+1}} = 0$ by Lemma 8.5(a). Similarly, if $i_t = e$ then $\Delta_{i_{t+1}} g_{p-1} = \Delta_{i_{t+1}}$. If no such t exists, then $i_1 = j_1, \ldots, i_n = j_n$ and so

$$v_{\alpha,i} v^*_{\beta,j} = G_\alpha \gamma^n(g_z) G_\beta = \gamma^n(g_v) \gamma^n(g_z) \gamma^n(g_w)$$

where $z = i_n$ if $i_n \neq e$, and $z = p - 1$, if $i_n = e$.

But $i_n \leq d(v)$, and so $g_v g_z = g_v$. But by Lemma 8.4(e) $g_v g_w = 0$.

Now suppose that $n \neq k$, with $n > k$. Note that

$$d(w) = d(v) + 2(n - k).$$

If there is a $t < k$, such that $i_1 = j_1, \ldots, i_t = j_t$, and $i_{t+1} \neq j_{t+1}$ then as before we have

$$v_{\alpha,i} v^*_{\beta,j} = G_\alpha \gamma^{n-1}(\Delta_{i_n}) \ldots \gamma^t(\Delta_{i_{t+1}}) \gamma^t(g_z) \gamma^t(\Delta_{j_{t+1}})^* \ldots \gamma^{k-1}(\Delta_{j_k})^* G_\beta$$

where $z = i_t$ if $i_t \neq e$, $z = p - 1$ otherwise. But $z \leq i_{t+1} + 1$,

and so $\Delta_{i_{t+1}} g_z = \Delta_{i_{t+1}}$, then by Lemma 8.5(a) $v_{\alpha,i} v^*_{\beta,j} = 0$, since $i_{t+1} \neq j_{t+1}$.

Finally if $i_1 = j_1, \ldots, i_k = j_k$, then we have

$$\begin{aligned} v_{\alpha,i} v^*_{\beta,j} &= G_\alpha \gamma^{n-1}(\Delta_{i_n}) \ldots \gamma^k(\Delta_{i_{k+1}}) \gamma^k(g_z) G_\beta \\ &= \gamma^k[\gamma^{n-k}(g_v) \gamma^{n-k-1}(\Delta_{i_n}) \ldots \Delta_{i_{k+1}} g_z g_w] \end{aligned}$$

where $z = [i_t]$. But $z \leq d(w)$, and so $g_z g_w = g_w$. Note also that by definition of $i = (i_1, \ldots, i_n)$, we have

$$i_{k+1} \le d(v) + n - k - 1,$$

and so

$$i_{k+1} + 1 \;\le\; d(v) + n - k \;<\; d(v) + 2(n-k) = d(w).$$

But this implies that $u_{i_{k+1}} g_w = 0$ by Lemma 8.5(e). Hence $\Delta_{i_{k+1}} g_w = 0$,

and so

$$v_{\alpha,i}\, v^*_{\beta,j} = 0.$$

Lemma 8.7.

(a) $G^i_{(0,m)} = G^i_{(1,m+1)}, \quad i \in I_{(0,m)} \subseteq I_{(1,m+1)}.$

(b) $G^i_{(v,m)} = G^{(i,v-1)}_{(v-1,m+1)} + G^i_{(v+1,m+1)}, \; i \in I_{(v,m)}, \; v \in \hat{T}^{(0)}_{p,2,r-1},$

$v \neq 0, p-1, \bar{p}$.

(c) $G^i_{(\bar{p},m)} = G^{(i,\varepsilon)}_{(p-1,m+1)}, \quad i \in I_{(\bar{p},m)}.$

(d) $G^i_{(p-1,m)} = G^{(i,v-1)}_{(p-2,m+1)} + G^i_{(\bar{p},m+1)} + G^i_{(p,m+1)}.$

(e) When $\beta = \| T_{p,2,r} \|$, and $r < \infty$, $G_{(p+r-2,p+r-2)} = G^{(p+r-3)}_{(p+r-3,p+r-1)} + g_{p+r-1}$, and for $m > p + r - 2$, $G^i_{(p+v-2,m)} = G^{(i,p+r-3)}_{(p+r-3,m+1)}, \quad i \in I_{(p+r-2,m)}.$

If there exists a faithful trace satisfying (8.24), then $g_{p+r-1} = 0$.

Proof:

For (a) note that $g_0 = g_1$, and if $\alpha = (0,m) \in \hat{T}^{(0)}_{p,2,r}$ then m is even. Thus

$$G_{(0,m)} = \gamma^{m/2}(g_0) = \gamma^{m/2}(g_1) = G_{(1,m+1)}$$

and so (a) follows.

For (b) note that when $v \neq 0, p-1, \bar{p}, (p + r - 2$ if $r < \infty)$, then by Lemma 8.5(f)

$$g_v = g_{v+1} + (\beta\phi_{v-1}/\phi_v)\, g_v\, e_v\, g_v = g_{v+1} + \Delta^*_{v-1}\, \gamma(g_{v-1})\, \Delta_{v-1}.$$

Then since $G_{(v,m)} = \gamma^n(g_v)$ where $n = (m - v)/2$, we have

$$G_{(v,m)} = \gamma^n(g_{v+1}) + \gamma^n(\Delta_{v-1})^*\, \gamma^{n+1}(g_{v-1})\, \gamma^n(\Delta_{v-1}),$$

and so

$$G^i_{(v,m)} = \Delta^*_{i_1} \dots \gamma^{n-1}(\Delta_{i_n})^*\, \gamma^n(g_{v+1})\, \gamma^{n-1}(\Delta_{i_n}) \dots \Delta_{i_1}$$

$$+ \Delta^*_{i_1} \dots \gamma^{n-1}(\Delta_{i_n})^*\, \gamma^n(\Delta_{v-1})^*\, \gamma^{n+1}(g_{v-1})\, \gamma^n(\Delta_{v-1})\, \gamma^{n-1}(\Delta_{v-1}) \dots \Delta_{i_1}.$$

But $\gamma^n(g_{v+1}) = G_{(v+1,m+1)}$, and $\gamma^{n+1}(g_{v-1}) = G_{(v-1,m+1)}$, and so (b) follows.

By Lemma 8.5(g), we have $g_{\bar{p}} = \Delta^*_\varepsilon\, \gamma(g_{p-1})\, \Delta_\varepsilon$, and so if $n = (m - p)/2$, we have

$$G_{(\bar{p},m)} = \gamma^n(g_{\bar{p}}) = \gamma^n(\Delta_\varepsilon)^*\, \gamma^{n+1}(g_{p-1})\, \gamma^n(\Delta_\varepsilon),$$

but $\gamma^{n+1}(g_{p-1}) = G_{(p-1,m+1)}$, and so (c) follows.

For (d) we use Lemma 8.5(f) to obtain

$$g_{p-1} = g_p + (\beta\phi_{p-2}/\phi_{p-1})\, g_{p-1}\, e_{p-1}\, g_{p-1} + g_{\bar{p}} = g_p + \Delta^*_{p-2}\, \gamma(g_{p-2})\, \Delta_{p-2} + g_{\bar{p}}$$

Hence if $n = (m - (p - 1))/2$, we have

$$\begin{aligned} G_{(p-1,m)} &= \gamma^n(g_{p-1}) \\ &= \gamma^n(g_p) + \gamma^n(\Delta_{p-2})^*\, \gamma^{n+1}(g_{p-2})\, \gamma^n(\Delta_{p-2}) + \gamma^n(g_{\bar{p}}) \\ &= G_{(p,m+1)} + \gamma^n(\Delta_{p-2})^*\, G_{(p-2,m+1)}\, \gamma^n(\Delta_{p-2}) + G_{(\bar{p},m+1)} \end{aligned}$$

and (d) follows.

(e) If $\beta = ||T_{p,2,r}||$, Then $\varphi_{p+r-1}(\beta) = 0$. Then putting $t = p + r - 1$, it follows by Lemma 8.4(b) that

$$e_t\, g_t\, (e_t\, g_t)^* = e_t\, g_t\, e_t = (\phi_t/\beta\phi_{t-1})\, e_t\, g_{t-1} = 0,$$

and so $e_t\, g_t = 0$. But then we have

$$e_{t+1}\, g_t = \beta^2\, e_{t+1}\, e_t\, e_{t+1}\, g_t = \beta^2\, e_{t+1}(e_t\, g_t)\, e_{t+1} = 0$$

and by induction $e_k\, g_t = 0$ for all $k \geq t$, and thus for all k.

It follows that $\gamma^n(g_t) = 0$ for all $n \geq 1$. Then from (8.16) and Lemma 8.5 we have

$$g_{t-1} = g_t + \Delta^*_{t-2}\, \gamma(g_{t-2})\, \Delta_{t-2},$$

from which we obtain (e) by applying γ.

If a faithful trace, tr satisfying (8.24) exists, then since $\phi_t(\beta) = 0$,

$$\begin{aligned} \mathrm{tr}(g_t) &= \mathrm{tr}(g_{t-1}) - (\beta\phi_{t-2}/\phi_{t-1})\, \mathrm{tr}(g_{t-1}\, e_{t-1}) \\ &= (1 - (\phi_{t-2}/\beta\phi_{t-1}))\, \mathrm{tr}(g_{t-1}) = (\phi_t/\beta\phi_{t-1})\, \mathrm{tr}(g_{t-1}) = 0. \end{aligned}$$

Hence $g_{p+r-1} = 0$.

Lemma 8.8.

Suppose that (8.62) holds for $r = \infty$, then we have for each $m \geq 0$

(8.67) $$1 = \sum_v G^i_{(v,m)}$$

where the summation is over all vertices (v,m) on level m of $\hat{T}^{(0)}_{p,2,r}$ and all $i \in I_{(v,m)}$. If (8.62) holds for some $r < \infty$, then (8.67) is true for $m \leq p + r - 2$, and for each $m > p + r - 2$ we have

(8.68) $$1 = \sum G^i_{(v,m)} + g_{p+r-1}.$$

Proof:

We use the splitting rules for $G^i_{(v,m)}$ in Lemma 8.7, c.f. [1] is an order unit for $K_0(A(T_{p,2,r}))$. in section 3.

Let $m \geq 1$, $p \geq 2$, and $r \in \{2,3,\ldots,\infty\}$. Let $\alpha = (v, m+1) \in \hat{T}^{(0)}_{p,2,r}$, with $d(v) < m+1$. Put $n = (m+1-d(v))/2$. Note that for such α, we have $\alpha' = (v, m-1) \in \hat{T}^{(0)}_{p,2,r}$. Denote by $\bar{I}_\alpha$ the set

$$\bar{I}_\alpha = \{i \in I_\alpha;\ (i_1, i_2, \ldots, i_{n-1}) \in I_{\alpha'}\}.$$

For example, if $v \neq 0, p-1, \bar{p}$ and if $r < \infty$, $v \neq p+r-2$, then $\bar{I}_\alpha$ consists of all $I \in I_\alpha$ with $i_n = v-1$, or $i_n = v$ and $i_{n-1} \leq v$. Suppose also $\phi_v(\beta) > 0$, for all $v \in \hat{T}^{(0)}_{p,2,r}$.

Lemma 8.9.

(a) Let $t \in \{1,2,\ldots,p-2,\varepsilon,p-1,\ldots\}$, and $s \geq 1$, then if $m \geq t+2s+3$ we have

$$\gamma^s(\Delta_t)\, e_m = e_m\, \gamma^s(\Delta_t).$$

(b) Let $i = (i_1, \ldots, i_n) \in I_\alpha$, then we have

$$i_k \leq i_{n-t} + (n-k-t)$$

for $t = 0,1,\ldots,n-1$, and $k = 1,2,\ldots,n-t$.

(c) For $v \neq 0, \bar{p}$ and if $r < \infty$, $v \neq p+r-2$, we have

$$\gamma(g_v)\, \Delta_v\, e_{v+1} = (\phi_{v+1}/\beta\phi_v)^{\frac{1}{2}} \beta^v\, e_1 e_2 \ldots e_{v+1} g_v.$$

(d) For $v \neq 0, 1, \bar{p}$ we have

$$\gamma(g_v)\, \Delta_{v-1}\, e_{v+1} = (\phi_{v-1}/\beta\phi_v)^{\frac{1}{2}} \beta^v\, e_1 e_2 \ldots e_{v+1} g_v.$$

and when $v = 1$ we have

$$\gamma(g_1) e_2 = (\phi_0/\beta\phi_1)^{\frac{1}{2}}\, \beta e_1 e_2 g_1.$$

(e) For $v \neq 0$, and if $r < \infty$, $v \neq p+r-2$, we have

$$\gamma(\Delta_v)\, \Delta_{v+1} e_{v+3} = 0.$$

(g) $\gamma(g_{\bar{p}})\, \Delta_\varepsilon\, e_{p+1} = \beta^p\, e_1 e_2 \ldots e_{p+1}\, g_{\bar{p}}$.

(g) $\gamma(g_{p-1})\,\Delta_\varepsilon\, e_p = \beta^{p-1}\,(\phi_{\bar{p}}/\beta\phi_{p-1})^{\frac{1}{2}}\, e_1 e_2 \cdots e_p g_{p-1}$.

(h) $G_\alpha\, e_m = 0$.

Proof:

(a) First note that $e_m \Delta_r = \Delta_r\, e_m$ for $m \ge r + 3$, and $r \ne \varepsilon$. Also $e_m \Delta_\varepsilon = \Delta_\varepsilon e_m$, for $m \ge p + 2$, i.e. $m \ge \varepsilon + 3$. Then since $m = k + 2s$, with $k \ge t + 3$, we have

$$e_m\, \gamma^s(\Delta_t) = \gamma^s(e_k)\, \gamma^s(\Delta_t) = \gamma^s(e_k \Delta_t) = \gamma^s(\Delta_t e_k) = \gamma^s(\Delta_t) e_m.$$

(b) This is clear from the definition of I_α.

(c) By Lemma 8.4(a),(b) we have

$$\Delta_v\, e_{v+1} = \frac{\beta\phi_v}{\sqrt{(\phi_{v-1}\phi_{v+1})}}\, e_2\, g_2\, e_3\, g_3 \cdots e_{v+1} g_{v+1} e_{v+1}$$

$$= \beta^v\, (\phi_1\phi_v/\phi_0\phi_{v+1})^{\frac{1}{2}}\, e_2\, g_2 \cdots e_v\, g_v \cdot \frac{\phi_{v+1}}{\beta\phi_v} e_{v+1}\, g_v$$

$$= \beta^v\, (\phi_1\phi_v/\phi_0\phi_{v+1})^{\frac{1}{2}}\, \frac{\phi_{v+1}}{\beta\phi_v}\, e_2\, e_3 \cdots e_v\, e_{v+1}\, g_v$$

$$= \beta^v (\phi_{v+1}/\beta\phi_v)^{\frac{1}{2}}\, e_2\, e_3 \cdots e_{v+1}\, g_v.$$

Hence by Lemma 8.3 we have

$$\gamma(g_v)\,\Delta_v\, e_{v+1} = \beta^{2v}\, e_1 e_2 \cdots e_{v+1}\, g_v\, e_{v+1} \cdots e_1\, \beta^v (\phi_{v+1}/\beta\phi_v)^{\frac{1}{2}}\, e_2\, e_3 \cdots e_{v+1}\, g_v$$

$$= \beta^v\, (\phi_{v+1}/\beta\phi_v)^{\frac{1}{2}}\, e_1 e_2 \cdots e_{v+1}\, g_v\, (\beta^{2v}\; e_{v+1} \cdots e_2\, e_1\, e_2 \cdots e_{v+1})\, g_v$$

$$= \beta^v\, (\phi_{v+1}/\beta\phi_v)^{\frac{1}{2}}\, e_1 e_2 \cdots e_{v+1}\, g_v\, e_{v+1}\, g_v$$

and so (c) follows using Lemma 8.4.

(d) Using Lemma 8.3, and Lemma 8.4(a), (e) and (8.46) we have

$$\gamma(g_v)\,\Delta_v\, e_{v+1} = \beta^{2v}\, \beta^{v-1}\, (\phi_1\phi_{v-1}/\phi_0\phi_v)^{\frac{1}{2}}\, e_1\, e_2 \cdots e_{v+1}\, g_v\, e_{v+1} \cdots$$

$$e_1\, e_2\, g_2 \cdots e_v\, g_v\, e_{v+1}$$

$$= \beta^{v-1}\, (\beta\phi_{v-1}/\phi_v)^{\frac{1}{2}}\; e_1\, e_2 \cdots e_{v+1}\, g_v\, (\beta^{2v} e_{v+1} \cdots e_1\, e_2\, e_1 \cdots e_{v+1})\, g_v$$

$$= \beta^{v-1}\, (\beta\phi_{v-1}/\phi_v)^{\frac{1}{2}}\; e_1\, e_2 \cdots e_{v+1}\, g_v\, e_{v+1}\, g_v\, .$$

But $\beta^{v-1}(\beta\phi_{v-1}/\phi_v)^{\frac{1}{2}} = \beta^{v}(\phi_{v-1}/\beta\phi_v)^{\frac{1}{2}}$, and $e_{v+1}\, g_v\, e_{v+1}\, g_v = e_{v+1}\, g_v$, and so we have the first part of (d). Also

$$\gamma(g_1)\, e_2 = \beta^2 e_1 e_2\, g_1 e_2\, e_1 e_2 = e_1 e_2\, g_1 = \beta\, \sqrt{\frac{\phi_0}{\beta\phi_1}}\, e_1\, e_2\, g_1 .$$

(e) First note that

$$\gamma(\Delta_v) = \beta^{2(v+2)} e_1\, e_2 \cdots e_{v+3}\, \Delta_v\, e_{v+3} \cdots e_2\, e_1 ,$$

and so, using Lemma 8.4(a), (e) and (8.46) we have

$$\gamma(\Delta_v)\, \Delta_{v+1} e_{v+3} = \delta e_1 \cdots e_{v+3}\, \Delta_v\, e_{v+3} \cdots e_1\, e_2\, g_2\, e_3\, g_3 \cdots e_{v+2}\, g_{v+2}\, e_{v+3}$$

$$= \delta e_1 \cdots e_{v+3}\, \Delta_v\, e_{v+3} \cdots e_2\, e_1\, e_2 \cdots e_{v+3}\, g_{v+2}$$

$$= \delta' e_1 \cdots e_{v+3}\, \Delta_v\, e_{v+3}\, g_{v+2}$$

where δ, δ' are scalars. But $\Delta_v = u_1\, u_2 \ldots u_v$, and

$$u_v\, e_{v+3}\, g_{v+2} = \lambda\, e_{v+1}\, g_{v+1}\, e_{v+3}\, g_{v+2} = \lambda\, e_{v+1}\, e_{v+3}\, g_{v+2} = \lambda\, e_{v+3}\, e_{v+1}\, g_{v+2} = 0$$

where λ is a scalar, by Lemma 8.4(a), (c) and (e), and so $\gamma(\Delta_v)\, \Delta_{v+1} e_{v+3} = 0$.

(f) As in (d), we have

$$\gamma(g_{\bar p})\, \Delta_\varepsilon\, e_{p+1}$$

$$= \beta^{2p}\, \beta^{p-1} (\phi_1 \phi_{p-1}/\phi_0 \phi_{\bar p})^{\frac{1}{2}}\; e_1\, e_2 \cdots e_{p+1}\, g_{\bar p}\, e_{p+1} \cdots e_1\, e_2\, g_2\, e_3\, g_3 \cdots e_p\, g_{\bar p}\, e_{p+1}$$

$$= \beta^{p}\, (\phi_{p-1}/\beta\phi_{\bar p})^{\frac{1}{2}}\, e_1\, e_2 \cdots e_{p+1}\, g_{\bar p}\, (\beta^{2p}\, e_{p+1} \cdots e_2\, e_1\, e_2 \cdots e_{p+1})\, g_{\bar p}$$

$$= \beta^{p}\, (\phi_{p-1}/\beta\phi_{\bar p})^{\frac{1}{2}}\; e_1 e_2 \cdots e_{p+1}\, g_{\bar p}\; e_{p+1}\, g_{p} .$$

But $\phi_{p-1} = \beta\phi_{\bar p}$, and so (f) follows.

(g) Since $e_p\, g_{\bar p}\, e_p = (\phi_{\bar p}/\beta\phi_{p-1})\ e_p\, g_{p-1}$, we have

$$\gamma(g_{p-1})\,\Delta_\varepsilon\, e_p$$

$$= \beta^{2(p-1)}\, e_1 e_2 \cdots e_p\, g_{p-1}\, e_p \cdots e_1\, \beta^{p-1}\, (\phi_1\phi_{p-1}/\phi_0\phi_{\bar p})^{\frac{1}{2}}\, e_2\, g_2 \cdots e_{p-1}\, g_{p-1}\, e_p\, g_{\bar p}\, e_p$$

$$= \beta^{p-1}\, (\beta\phi_{p-1}/\phi_{\bar p})^{\frac{1}{2}}\, e_1 e_2 \cdots e_p\, g_{p-1}\, (\beta^{2(p-1)}\, e_p \cdots e_2 e_1 e_2 \cdots e_p)\, (\phi_{\bar p}/\beta\phi_{p-1})^{\frac{1}{2}}\, g_{p-1}$$

$$= \beta^{p-1}\, (\beta\phi_{p-1}/\phi_{\bar p})^{\frac{1}{2}}\, e_1 \cdots e_p\, g_{p-1}\, e_p\, (\phi_{\bar p}/\beta\phi_{p-1})^{\frac{1}{2}}\, g_{p-1}$$

$$= \beta^{p-1}\, (\phi_{\bar p}/\beta\phi_{p-1})^{\frac{1}{2}}\, e_1 e_2 \cdots e_p\, e_{p-1}.$$

(h) Since $m = 2n + v - 1$, we have by Lemma 8.4(c) that

$$G_\alpha e_m = \gamma^n(g_v)\,\gamma^n(e_{v-1}) = \gamma^n(g_v\, e_{v-1}) = 0.$$

Proposition 8.10.

If $\alpha = (v, m+1) \in \hat{\Gamma}^{(0)}_{p,2,r}$, $i,j \in I_\alpha$:

$$G^i_\alpha\, e_m\, G^j_\alpha = \gamma^{ij}_\alpha\, G^{ij}_\alpha$$

where if $i,j \in \bar{I}_\alpha$ we have

$$\gamma^{ij}_\alpha = \begin{cases} \dfrac{\sqrt{\phi_{\omega(i)}\phi_{\omega(j)}}}{\beta\phi_v}\,\delta_{i_1 j_1}\cdots\delta_{i_{n-1} j_{n-1}}, & v \neq 0, p-1, \bar p, \text{ and if } r < \infty,\ v \neq p + r - 2 \\[2ex] \dfrac{\sqrt{\phi_{\zeta(i)}\phi_{\zeta(j)}}}{\beta\phi_{p-1}}\,\delta_{i_1 j_1}\cdots\delta_{i_{n-1} j_{n-1}}, & v = p - 1 \\[2ex] 1\,\delta_{i_1 j_1}\cdots\delta_{i_{n-1} j_{n-1}}, & v = 0, \bar p, \text{ and if } r < \infty,\ v = p + r - 2. \end{cases}$$

and if i, or $j \notin \bar{I}_\alpha$, then $\gamma^{ij}_\alpha = 0$. Here we have

$$\omega(i) = \begin{cases} v - 1 & \text{if} \quad i_n = v - 1 \\ v + 1 & \text{if} \quad i_n = v \end{cases}$$

and

$$\zeta(i) = \begin{cases} p - 2 & \text{if} \quad i_n = p - 2 \\ \bar{p} & \text{if} \quad i_n = \varepsilon = p - 2 + \frac{1}{2} \\ p & \text{if} \quad i_n = p - 1. \end{cases}$$

Moreover if $\beta = (w, m+1) \in \hat{T}^{(0)}_{p,2,r}$, with $w \neq v$, then

$$G^i_\alpha e_m G^j_\beta = 0$$

for all $i \in I_\alpha$, $j \in I_\beta$. Finally we have

$$G_{(m+1,m+1)} e_m = 0.$$

Proof:

First suppose that $v \neq 0, \bar{p}, p-1$, or if $r < \infty$, $v \neq p + r - 2$. Let $i \in I_\alpha$, then $i_{n-1} \leq v$, and so by Lemma 8.9(b) we have $i_k \leq v + n + k - 1$, for $k = 1, 2, \ldots, n-1$.

Then since

$$i_k + 2(k-1) + 3 \leq v + 2n - 1 = m$$

for $k = 1, 2, \ldots, n-1$, it follows from Lemma 8.9(a) that

$$v_{\alpha,i} e_m = G_\alpha \gamma^{n-1}(\Delta_{i_n}) \ldots \Delta_{i_1} e_m = G_\alpha \gamma^{n-1}(\Delta_{i_n}) e_m \gamma^{n-2}(\Delta_{i_{n-1}}) \ldots \Delta_{i_1}.$$

But $e_m = \gamma^{n-1}(e_{v+1})$, and so by Lemma 8.9(c) or (d) we have

$$G_\alpha \gamma^{n-1}(\Delta_{i_n}) e_m = \gamma^{n-1}(\gamma(g_v) \Delta_{i_n} e_{v+1})$$

$$= (\phi_{\omega(i_n)}/\beta\phi_v)^{\frac{1}{2}} \beta^v \gamma^{n-1}(e_1 e_2 \ldots e_{v+1} g_v).$$

Now, it follows from the proof of Lemma 8.6(b) that for $j \in I_\alpha$

$$V_{\alpha,i} V^*_{\alpha,j} = G_\alpha \gamma^{n-1}(\Delta_{i_n}) e_m \gamma^{n-2}(\Delta_{i_{n-1}}) \ldots \Delta_{i_1} \Delta^*_{j_n} \ldots \gamma^{n-2}(\Delta_{j_{n-1}})^* e_m \gamma^{n-1}(\Delta_{j_n})^* G_\alpha$$

vanishes if $(i_1, i_2, \ldots, i_{n-1}) \neq (j_1, j_2, \ldots, j_{n-1})$, otherwise using (8.66)we have

$$V_{\alpha,i}V^*_{\alpha,j} = \gamma^{n-1}(\gamma(g_v)\,\Delta_{i_n}\,e_{v+1})\,\gamma^{n-1}(g_{[i_{n-1}]})\,\gamma^{n-1}(e_{v+1}\,\Delta^*_{j_n}\,\gamma(g_v))$$

$$= \frac{(\phi_{\omega(i_n)}\phi_{\omega(j_n)})^{\frac{1}{2}}}{\beta\phi_v}\,\gamma^{n-1}(\beta^{2v}e_1e_2\cdots e_{v+1}\,g_v\,g_{[i_{n-1}]}\,g_v\,e_{v+1}\cdots e_1).$$

But $g_{[i_{n-1}]}\,g_v = g_v$, and so by Lemma 8.3

$$V_{\alpha,i}V^*_{\alpha,j} = \frac{(\phi_{\omega(i_n)}\phi_{\omega(j_n)})^{\frac{1}{2}}}{\beta\phi_v}\,\gamma^{n-1}(\gamma(g_v)) = \gamma^{ij}_{\alpha}\,\gamma^{n}(g_v) = \gamma^{ij}_{\alpha}\,G_{\alpha}.$$

Hence $G^{i}_{\alpha}\,e_m\,G^{j}_{\alpha} = \gamma^{ij}_{\alpha}\,G^{ij}_{\alpha}$.

Now suppose that $i \notin I_{\alpha}$, then either $i_n < v - 1$, or $i_{n-1} = v + 1$. If $i_n < v - 1$, then by Lemma 8.9(b), $i_k \le v + n - k - 2$, for $k = 1,2, \dots, n$. Then since $i_k + 2(k-1) + 3 \le v + 2n - 1 = m$, for $k = 1, \dots, n$ we have

$$v_{\alpha,i}\,e_m = G_{\alpha}\,e_m\,\gamma^{n-1}(\Delta_{i_n})\cdots\Delta_{i_1}$$

but this vanishes by Lemma 8.9(h). If $i_{n-1} = v + 1$, then $i_k + 2(k-1) + 3 \le m$ for $k = 1,2, \dots, n-2$, and so

$$v_{\alpha,i}\,e_m = G_{\alpha}\,\gamma^{n-1}(\Delta_{i_n})\,\gamma^{n-2}(\Delta_{i_{n-1}})\,e_m\,\gamma^{n-3}(\Delta_{i_{n-2}})\cdots\Delta_{i_1}.$$

Now if $i_{n-1} = v + 1$, then $i_n = v$, thus since $e_m = \gamma^{n-2}(e_{v+3})$ we have

$$G_{\alpha}\,\gamma^{n-1}(\Delta_{i_n})\,\gamma^{n-2}(\Delta_{i_{n-1}}) = \gamma^{n-2}(\gamma^2(g_v)\,\gamma(\Delta_v)\,\Delta_{v+1}\,e_{v+3})$$

and so it follows from Lemma 8.9(e) that $v_{\alpha,i}\,e_m = 0$.

The proof that $G^{i}_{\alpha}\,e_m\,G^{j}_{\alpha} = \gamma^{ij}_{\alpha}\,G^{ij}_{\alpha}$, for $\alpha = (v, m+1)$, with $v = 0, p-1, \tilde{p}$, or $p + r - 2$ when $r < \infty$, is similar, using Lemma 8.9(f) and (g) for example.

The proof that $G^i_\alpha e_m G^j_\beta = 0$ for $\alpha \neq \beta$, is essentially a combination of the above proof, and that of Lemma 8.6(c). Thus if $i \notin I_\alpha$, then $G^i_\alpha e_m = 0$, otherwise we proceed as above to get

$$v_{\alpha,i} e_m v^*_{\beta,j}$$
$$= G_\alpha \gamma^{n-1}(\Delta_{i_n}) \dots \gamma^{k-1}(\Delta_{i_k}) e_m \gamma^{k-2}(\Delta_{i_{k-1}}) \dots \Delta_{i_1}.\Delta^*_{j_1} \dots \gamma^{t-2}(\Delta_{j_{t-1}})^* e_m$$
$$\gamma^{t-1}(\Delta_{i_t})^* \dots \gamma^{s-1}(\Delta_{i_s})^* G_\beta,$$

and then either

$$\gamma^{k-2}(\Delta_{i_{k-1}}) \dots \Delta_{i_1}.\Delta^*_{j_1} \dots \gamma^{t-2}(\Delta_{j_{t-1}})^* = 0$$

or more detailed arguments are necessary as in the proof of Lemma 8.6(c).

Finally we have

$$G_{(m+1,m+1)} e_m = g_{m+1} e_m = 0.$$

Lemma 8.11.

For $m \geq 1$, we have

$$e_m = \sum \gamma^{ij}_\alpha G^{ij}_\alpha$$

where the summation is over all vertices α on level $m+1$ of $\hat{T}_{p,2,r}$, and all $i,j \in I_\alpha$, and the coefficients $\gamma_{ij} \in \mathbb{C}$ are given in Proposition 8.10.

Proof:

By Lemma 8.8 we have $1 = \sum G^i_\alpha + u$, where we can take $u = 0$, if $r = \infty$, or if $m \leq p + r - 3$, otherwise note that $ue_k = 0$ for all k, and the summation is over all vertices α on level $m+1$ of $\hat{T}_{p,2,r}$, and $i \in I_\alpha$. It follows using Proposition 8.10 that

$$e_m = 1\, e_m 1 = (\Sigma G^i_\alpha + u)\, e_m\, (\Sigma G^j_\alpha + u) = \Sigma\, G^i_\alpha\, e_m\, G^j_\alpha = \Sigma\, \gamma^{ij}_\alpha\, G^{ij}_\alpha.$$

Remark 8.12.

It follows immediately that G^i_α is a minimal idempotent in $A(\tau,p)_{m+1}$ for each $\alpha = (v,m+1)$, on level $m+1$ of $\hat{T}_{p,2,r}$, and $i \in I_\alpha$.

Lemma 8.13.

Let $p \geq 2$, $\tau > 0$, and $e_1, e_2, \dots, e_{\bar{p}}$ be a sequence of projections satisfying the relations (8.17) - (8.21). If $\tau = ||A_{p+1}||^{-2}$, then $A(\tau,p) \cong A(\tau)$, the Jones algebra with parameter τ, otherwise $A(\tau,p)$ is trivial unless

$$\beta = \tau^{-\frac{1}{2}} \in \{||T_{p,2,r}||;\, r \geq 2\} \cup [\,||T_{p,2,\infty}||, \infty).$$

Proof:

First we recall from Lemma 3.4 the following facts. Let $(\Gamma_r)^\infty_{r=1}$ be the sequence of subgraphs of $T_{p,2,\infty}$ given by: $\Gamma_1 = A_{p+1}$, consisting of the vertices $0,1,\dots,p-1,\bar{p}$ and edges between them, and for $r \geq 2$, $\Gamma_r = T_{p,2,r}$, consisting of the vertices $0,1,\dots,p-1,\bar{p},p,\dots,p+r-2$. Put $||\Gamma_r|| = \beta_r$. Then β_r is a strictly increasing sequence converging to $||T_{p,2,\infty}||$. Note that if $\beta_r < 2$, then $\beta_r = 2\cos(\pi/m)$, for some $m \geq 3$. Now β_r is the largest zero of ϕ_{p+r-1}, and if β'_r denotes the second largest zero of ϕ_{p+r-1} for $r \geq 2$, then we have the interlacing property

$$\beta'_{r+1} < \beta_r < \beta_{r+1}$$

for all $r \geq 1$. Note also that $\phi_v(\beta_r) > 0$ for all $v \in \Gamma^{(0)}_r$. Thus if $\beta_r < \beta < \beta_{r+1}$, then $\phi_{p+r-1}(\beta)/\beta\phi_{p+r-2}(\beta) < 0$, for $r \geq 1$. Finally note that if $\beta \geq \beta_r$, then $g_v = g_v(\beta)$ is defined for all $v \in \Gamma^{(0)}_{r+1}$.

We can clearly assume that if $\beta < 2$, then $\beta = 2\cos(\pi/m)$ for some $m \geq 3$. We first show that $\beta \leq \beta_1$ is not allowed. Suppose that $\beta = \beta_1$, then using Lemma 8.4(b) we have

$$(g_p e_p)^*(g_p e_p) = e_p g_p e_p = (\phi_p/\beta\phi_{p-1})\ e_p g_{p-1} = 0$$

Hence $g_p e_p = 0$. Next, from (8.12) and (8.50) we have $f_p = g_p + e_{\bar{p}}$, where $f_p = 1 - e_1 \vee \dots \vee e_{p-1}$. Then since $\beta = \|A_{p+1}\|$, and $S_{p+1}(\beta) = 0$, we have $\beta S_{p-1}(\beta)/S_p(\beta) = \beta^2$. Thus

$$f_{p+1} = f_p - (\beta S_{p-1}/S_p)\ f_p e_p f_p = f_p - \beta^2 f_p e_p f_p = g_p + e_{\bar{p}} - \beta^2 (g_p + e_{\bar{p}})\, e_p (g_p + e_{\bar{p}})$$

$$= g_p + e_{\bar{p}} - \beta^2 e_{\bar{p}} e_p e_{\bar{p}} = g_p .$$

It follows that $e_{\bar{p}} = f_p - f_{p+1}$ is in the C*-algebra generated by $1, e_1, e_2, \dots, e_p$.

The only other cases we need to consider for $\beta < \beta_1$, are when $\beta = \|A_k\| = 2\cos(\pi/(k+1))$, $k = 3, \dots, p$. Then, if $f_k = 1 - e_1 \vee \dots \vee e_{k-1}$, since $S_k(\beta) = 0$, we have

$$e_k\, f_k\, e_k = (S_k/\beta S_{k-1})\ f_{k-1}\, e_k = 0,$$

and hence $f_k\, e_k = 0 = e_k\, f_k$. Then we have

$$0 = e_{k+1}\, f_k\, e_k\, e_{k+1} = f_k e_{k+1} e_k\, e_{k+1} = \beta^{-2}\, f_k\, e_{k+1}$$

and by induction $f_k\, e_\ell = 0$ for all $\ell \geq k$. It then follows that

$$1 - f_{k+1} = (1 - f_k) \vee e_k = e_k + (1 - f_k) - e_k(1 - f_k) = 1 - f_k,$$

and by induction that $1 - f_\ell = 1 - f_k$ for all $\ell \geq k$. In particular $f_p = f_{p+1}$. But $e_{\bar{p}} f_p = e_{\bar{p}}$, and so $e_{\bar{p}} \leq f_p = f_{p+1}$. It then follows that

$$(e_{\bar{p}}\, e_p)^*\, e_{\bar{p}}\, e_p = e_p\, e_{\bar{p}}\, e_p \ \leq\ e_p f_{p+1}\, e_p = 0$$

and so $e_{\bar{p}}\, e_p = 0$. Thus $e_{\bar{p}} = \beta^2 e_{\bar{p}}\, e_p\, e_{\bar{p}} = 0$.

Now suppose that $\beta_r < \beta < \beta_{r+1}$, for $r \geq 1$, then since $\phi_{p+r-1}/\beta\phi_{p+r-2} < 0$, by Lemma 8.4(b) we have, putting $t = p + r - 1$,

$$0 \le (g_t e_t g_t)^2 = (\phi_t/\beta\phi_{t-1})\ g_t e_t g_t = (\phi_t/\beta\phi_{t-1})\ (e_t g_t)^* (e_t g_t) \le 0,$$

and so $e_t g_t = 0$. Then using Lemma 8.4(b) again gives

$$0 = e_t g_t e_t = (\phi_t/\beta\phi_{t-1})\, e_t g_{t-1}\,,$$

and so $e_t g_{t-1} = 0$.

If $r = 1$, then we have by (8.51)

$$0 = e_p g_p = e_p(g_{p-1} - (\beta\phi_{p-2}/\phi_{p-1})\, g_{p-1} e_{p-1} g_{p-1} - g_{\bar p}) = -e_p g_{\bar p} = -e_p e_{\bar p}$$

and so $e_{\bar p} = \beta^2 e_{\bar p} e_p e_{\bar p} = 0$.

For $r \ge 2$, note first that $e_k g_t = 0$ for all k (see the proof of Proposition 8.7 (e)), and it is clear also that when $m + 1 \ge t$, we can write the identity as $1 = \Sigma G^{i}_{\alpha} + g_t$, where α runs over all vertices on level $m + 1$ of $\hat{T}_{p,2,r}$ and $i \in I_{\alpha}$. We now show that if $m = 2t$, then $G^{i}_{\alpha} = 0$ for all α on level $m + 1$ of $\hat{T}_{p,2,r}$, and all $i \in I_{\alpha}$. Note that for $\alpha = (v, 2t + 1) \in \hat{T}_{p,2,r}$, $d(v)$ is odd, and if $n = (2t + 1 - d(v))/2$, then we can assume $n \ge 2$. If $n = 2$, then $d(v) = 2t - 3 = 2(p + r - 1) - 3 \ge p + r - 1$, since $p,r \ge 2$, and so we can take $i = (i_1, i_2) \in I_{\alpha}$, with $i_1 = 0$, and $i_2 = t - 2$. Next we show that if $n > 2$, then we can choose $i \in I_{\alpha}$, with $i_1 = 0$, $i_2 = t - 2$. First note that by Lemma 8.9(b), $i_2 \le i_n + (n - 2)$, and that

$$i_n + (n - 2) = i_n + \frac{2t + 1 - d(v)}{2} - 2 = (i_n - d(v)/2 - \tfrac{1}{2}) + (t - 2).$$

Now if $v \ne \bar p$, $d(v) = v$ is odd, and so if we consider paths $i \in I_{\alpha}$, with $i_n = v$, then $i_n - d(v)/2 - \frac{1}{2} = v/2 - \frac{1}{2} \ge 0$, i.e. $i_n + (n - 2) \ge t - 2$. It follows that a path with $i_2 = t - 2$ is allowed. If $v = \bar p$, then $d(\bar p) = p$ is odd, and so $p \ge 3$. Thus taking $i \in I_{\alpha}$, with $i_n = p - 1$, we have $i_n - d(\bar p)/2 - \frac{1}{2} = p - 1 - p/2 - \frac{1}{2} = (p - 3)/2 \ge 0$. Then since $i_n + (n - 2) \ge t - 1$, we can choose $i \in I_{\alpha}$, with $i_2 = t - 2$.

Next note that $\gamma(\Delta_{t-2})\,\gamma(g_{t-1}) = \gamma(\Delta_{t-2}\, g_{t-1}) = \gamma(\Delta_{t-2})$. But $g_t e_t = 0$, and so $\gamma(g_{t-1}) = 0$, which means $\gamma(\Delta_{t-2}) = 0$. But if $i \in I_{\alpha}$, is chosen as above with

$i_2 = t - 2$, then it follows that $G_\alpha^i = 0$, and finally since G_α^j is equivalent to G_α^i for all $j \in I_\alpha$, that $G_\alpha^j = 0$ for all $j \in I_\alpha$. It follows that $1 = g_1$, and so $e_m = 0$ for all $m \geq 1$.

Lemma 8.14.

Let $\beta = \tau^{-1/2} = ||T_{p,2,r}||$, for some r, $1 \leq r \leq \infty$. Suppose that there exists a faithful trace tr, satisfying (8.24). Then we have

(a) $\mathrm{tr}\,(\gamma(x)) = \tau\,\mathrm{tr}(x)$

(b) $\mathrm{tr}\,(g_v) = Q_v(\tau)$, for $v \in T^{(0)}_{p,2,r}$

(c) $\mathrm{tr}\,(G_\alpha) = Q_\alpha(\tau)$, for $\alpha \in \hat{T}^{(0)}_{p,2,r}$

where Q_v, Q_α are as defined in (3.15) and (3.24).

Proof:

(c) For $x \in A(\tau,p)$, we have by Lemma 8.3 that

$$\gamma(x) = \tau^{-n}\, e_1 e_2 \cdots e_{n+1} x e_{n+1} \cdots e_2 e_1,$$

and so by (8.46), (8.24)

$$\begin{aligned}\mathrm{tr}\,(\gamma(x)) &= \tau^{-n}\ \mathrm{tr}\,(e_1 e_2 \cdots e_{n+1} x e_{n+1} \cdots e_2 e_1)\\ &= \tau^{-n}\,\mathrm{tr}\,(e_{n+1} \cdots e_2 e_1 e_2 \cdots e_{n+1} x)\\ &= \tau^{-n}\,\mathrm{tr}\,(\tau^{n}\, e_{n+1} x) = \mathrm{tr}\,(e_{n+1} x) = \tau\,\mathrm{tr}(x)\,.\end{aligned}$$

(b) Now $g_0 = g_1 = 1$, and $Q_0 = Q_1 = 1$, and so (b) is true for $v = 0,1$. For $v = 2, \ldots, p-1$, we have

$$g_v = g_{v-1} - (\beta\phi_{v-2}/\phi_{v-1})\, g_{v-1}\, e_{v-1}\, g_{v-1}$$

and so by (8.24), and noting that $\phi_v = \beta\phi_{v-1} - \phi_{v-2}$ we see

$$\begin{aligned}\mathrm{tr}\,(g_v) &= \mathrm{tr}\,(g_{v-1}) - (\beta\phi_{v-2}/\phi_{v-1})\,\mathrm{tr}\,(\, e_{v-1} g_{v-1})\\ &= (\,1 - (\phi_{v-2}/\beta\phi_{v-1})\,\mathrm{tr}\,(g_{v-1})\end{aligned}$$

$$= (\phi_v/\beta\phi_{v-1})\,\mathrm{tr}\,(g_{v-1}).$$

It follows that for $v = 2, \ldots, p-1$,

$$\mathrm{tr}\,(g_v) = \frac{\phi_v}{\beta\phi_{v-1}}\,\frac{\phi_{v-1}}{\beta\phi_{v-2}}\,\cdots\,\frac{\phi_1}{\beta\phi_0}\,\mathrm{tr}\,(g_0) = \frac{\phi_v}{\beta^v} = Q_v(\tau).$$

Next, by (8.20), (8.21), and (8.24), we have

$$\mathrm{tr}\,(e_{\bar p}) = \mathrm{tr}\,(g_{\bar p}) = \tau^{-1}\,\mathrm{tr}\,(e_{\bar p}\,e_p e_{\bar p}) = \tau^{-1}\,\mathrm{tr}\,(e_p e_{\bar p}\,e_p)$$

$$= \tau^{-1}\,\mathrm{tr}\,(\tau\, e_p\, g_{p-1}) = \mathrm{tr}\,(e_p\, g_{p-1})$$

$$= \tau\,\mathrm{tr}\,(g_{p-1}) = \tau\, Q_{p-1}(\tau) = Q_{\bar p}(\tau).$$

Then by (8.14), (8.24), and the facts that $\phi_{\bar p}/\beta\phi_{p-1} = \beta^{-2}$, and

$\beta\phi_{p-1} - \phi_{p-2} - \phi_{\bar p} = \phi_p$, we have

$$\mathrm{tr}\,(g_{\bar p}) = \mathrm{tr}\,(g_{p-1}) - (\beta\phi_{p-2}/\phi_{p-1})\,\mathrm{tr}\,(g_{p-1}\, e_{p-1}) - \mathrm{tr}\,(e_{\bar p})$$

$$= (1 - \frac{\phi_{p-2}}{\beta\phi_{p-1}} - \frac{1}{\beta^2})\,\mathrm{tr}\,(g_{p-1}) = \frac{\phi_p}{\beta\phi_{p-1}}\cdot\frac{\phi_{p-1}}{\beta^{p-1}} = Q_p\,(\tau).$$

For $v > p$, one shows that $\mathrm{tr}\,(g_v) = (\phi_v/\beta\phi_{v-1})\,\mathrm{tr}\,(g_{v-1})$, using (8.15), and (b) follows.

(c) Let $\alpha = (v,m)$, and $n = (m - d(v))/2$, then $G_\alpha = \gamma^n\,(g_v)$ and by (a) we have

$$\mathrm{tr}\,(\gamma^n\,(g_v)) = \tau^n\,\mathrm{tr}\,(g_v) = \tau^n\, Q_v(\tau) = Q_\alpha(\tau).$$

Proof of Theorem 8.1 continued:

For τ as in (8.23) choose the corresponding r, $2 \le r \le \infty$, and define a map $\psi : A(T_{p,2,r}) \oplus \mathbb{C}\,(1 - e_1 \vee \ldots \vee e_{p+r-2} \vee e_{\bar p}) \to A(\tau,p)$ as follows.

Put $q = 1 - e_1 \vee \ldots \vee e_{p+r-2} \vee e_{\bar p}$. For $\alpha \in \hat{T}^{(0)}_{p,2,r}$, and $i,j \in I_\alpha = \mathrm{Path}\,(*,\alpha)$

$$\psi\,(i,j) = G^{ij}_\alpha,\ \ \psi\,(q) = q.$$

It is clear from Lemma 8.7 that this map is well defined. From Lemma 8.6 we see that it defines a *-homomorphism and by Lemmas 8.8, and 8.11, it is surjective. It remains only to show that the map is injective under the stated conditions.

When $r < \infty$, $A(T_{p,2,r})$ is simple, and so ψ is injective. Suppose there exists a Markov trace. To show that ψ is injective in this case, it is enough to show that tr $(G_\alpha) > 0$ for each $\alpha \in \hat{T}^{(0)}_{p,2,r}$. But by Lemma 8.14(c) we have tr $(G_\alpha) = Q_\alpha(\tau)$, which we know is positive if $\tau^{-1} \geq \|T_{p,2,\infty}\|$, see section 3, or the beginning of the proof of Lemma 8.13.

Remark 8.15:

The method employed in the proof of Theorem 8.1 should also work for infinite graphs Γ of the type indicated by Figure 24. Here Γ is a tree, with an infinite branch which has attached to it a finite number of branches of length one, and a distinguished vertex $*$.

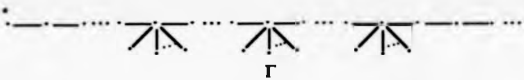

Figure 24

In these cases a presentation of $A(\Gamma)$ would be as follows. Let $\{e_v ; v \in \Gamma^{(0)}\}$ be a set of projections indexed by the vertices of Γ, such that the following relations are satisfied:

$$e_v e_w = e_w e_v, \quad d(v,w) \geq 2 \tag{8.69}$$

$$e_v e_w e_v = \tau e_v, \quad d(v,w) = 1, \qquad v,w \notin \partial\Gamma/\{*\}, \text{ or } \; v \in \partial\Gamma/\{*\} \text{ and } w \notin \partial\Gamma/\{*\} \tag{8.70}$$

$$e_v e_w e_v = \tau f_v e_v, \ d(v,w) = 1, \ v \notin \partial\Gamma/\{*\}, \ w \in \partial\Gamma/\{*\} \tag{8.71}$$

$$e_w e_v = 0, \ v,w \in \partial\Gamma/\{*\} \tag{8.72}$$

where $f_w = 1 - \bigvee_u e_u$, and the join is over all $u \in \Gamma^{(0)}$ such that $d(*,u) \le d(*,w) - 2$, and $\partial\Gamma$ denotes the boundary of Γ. This would include certain star shaped graphs considered in [HS].

Appendix A.

Let $\pi : G \to M_n$ be a unitary representation of a compact group G, and $\alpha_g = \overset{\infty}{\underset{1}{\otimes}} \mathrm{Ad}\,\pi(g)$, the product type action of G on the UHF algebra $A = \overset{\infty}{\underset{1}{\otimes}} M_n = \varinjlim A_m$, where $A_m = \overset{m}{\underset{1}{\otimes}} M_n$. The fixed point algebra A^G is AF, being the C*-inductive limit of the fixed point algebras A_m^G. Let $\{\chi_\alpha\}$ denote the irreducible characters of G, where χ_0 is the trivial character, and χ the character of π. For each $m = 0,1,2,\ldots,$ let

$$\chi^m = \sum_\alpha a_{m\alpha}\chi_\alpha \tag{A.1}$$

be the decomposition of the character χ^m, corresponding to the representation $\pi^{\otimes m}$ of G, into irreducible characters, where $a_{m\alpha}$ are positive integers. By Schur's lemma, the fixed point algebra A_m^G has the following decomposition into simple components

$$A_m^G \simeq \underset{\alpha}{\oplus} M_{a_{m\alpha}} \tag{A.2}$$

The multiplicity $k_{\alpha\beta}$ of the embedding of the simple component $M_{a_{m\alpha}}$ into $M_{a_{m+1,\beta}}$ is determined by the decomposition of $\chi\chi_\alpha$ into irreducible characters

$$\chi\chi_\alpha = \sum_\beta k_{\alpha\beta}\,\chi_\beta \tag{A.3}$$

To determine the Bratteli diagram, we only need to know (A.1) for $m = 1$ and (A.3). The representation graph [M] of G corresponding to π is the graph $\Gamma_\pi = \Gamma_\pi(G)$ with vertices $\{\chi_\alpha\}$ and $k_{\alpha\beta}$ edges between χ_α and χ_β. Then Γ_χ is connected if and only if π is faithful. It is clear that $A^G = A(\Gamma_\pi(G))$. Thus $A^G \simeq A(\Gamma_\chi)$, where the graph Γ_χ has distinguished vertex χ_0.

Let H be a closed subgroup of G. The embedding of A_m^G in A_m^H is determined as follows. Let $\{\sigma_\mu\}$ denote the irreducible characters of H, σ_0 the trivial representation and $\sigma = \chi | H$. Then

$$A_m^H \cong \oplus_\mu M_{b_{m\mu}} \tag{A.4}$$

if

$$\sigma^m = \sum_\mu b_{m\mu} \sigma_\mu \tag{A.5}$$

The multiplicity $q_{\alpha\mu}$ of the embedding of $M_{a_{m\alpha}}$ in $M_{b_{m\mu}}$ is given by the decomposition of $\chi_\alpha | H$ into irreducible characters of H:

$$\chi_\alpha | H = \sum_\mu q_{\alpha\mu} \sigma_\mu \tag{A.6}$$

with

$$q_{o\mu} = \delta_{o\mu}.$$

Let $G = SU(2)$, $\pi : G \to M_2$ the standard representation, $H = \mathbb{T}$, so that $(\pi | \mathbb{T})(z) = \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}$, $z \in \mathbb{T}$. Let $\{\chi_i\}_{i=0}^{\infty}$ be the irreducible characters of $SU(2)$, where χ_0 is the trivial character and χ_1 the character of π. Then by the Clebsch-Gordon formula we have

$$\chi_1 \chi_i = \chi_{i-1} + \chi_{i+1} \qquad i = 0,1,2,... \tag{A.7}$$

if $\chi_{-1} = 0$. Thus the representation graph Γ_χ is A_∞, with distinguished vertex $*$ $= 0$, and so $(\otimes M_2)^{SU(2)} \cong A(A_\infty)$ [BP,W], with Bratteli diagram as in Figure 25.

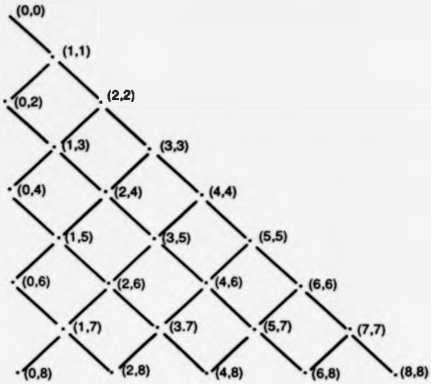

Figure 25

Let $\{\sigma_i\}_{i=-\infty}^{\infty}$ be the irreducible characters of $\mathbb{T}$, where $\sigma_i(z) = z^i$. Then if $\sigma = \chi_1|\mathbb{T}$ we have

$$\sigma = \sigma_1 + \sigma_{-1}$$

and

$$\sigma\sigma_i = \sigma_{i-1} + \sigma_{i+1}, \qquad i \in \mathbb{Z} \tag{A.8}$$

Thus Γ_σ is identified with $A_{\infty,\infty}$ as in Figure 26 with distinguished vertex $* = 0$, and so $(\otimes M_2)^{\mathbb{T}} \cong A(A_{\infty,\infty})$ [B], with Bratteli diagram as in Figure 27.

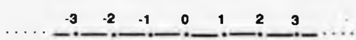

Figure 26 : $A_{\infty,\infty}$

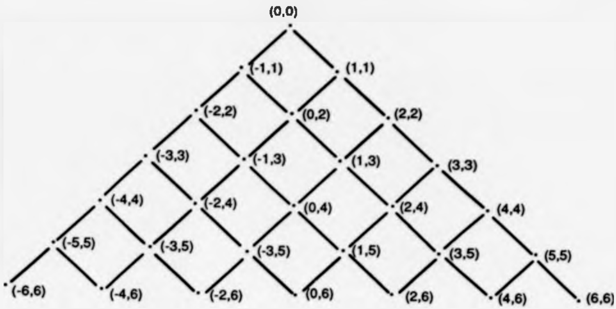

Figure 27

The embedding $A_m^{SU(2)} \to A_m^{\mathbb{T}}$ is determined by (A.6). We have $\chi_o | \mathbb{T} = \sigma_o$, $\chi_1 | \mathbb{T} = \sigma_1 + \sigma_{-1}$, and $\chi_2 | \mathbb{T} = (\chi_1^2 - \chi_o) | \mathbb{T} = (\sigma_1 + \sigma_{-1})^2 - \sigma_o = \sigma_2 + \sigma_o + \sigma_{-2}$. Inductively from

$$\chi_{i+1} = \chi_1 \chi_i - \chi_{i-1} \qquad \text{(A.9)}$$

we see

$$\chi_m | \mathbb{T} = \sigma_m + \sigma_{m-2} + \ldots + \sigma_{-m+2} + \sigma_{-m} \qquad \text{(A.10).}$$

From (A.9) we see that $\chi_i = S_i(x) \in \mathbb{Z}[x]$, if $x = \chi_1$, where $\{S_i\}$ are the Chebyshev polynomials of the second kind, the polynomials associated to the graph A_∞. Similarly the polynomials $\{\phi_i\}$ associated with the graph $A_{\infty,\infty}$ may be obtained from the irreducible characters of $\mathbb{T}$ by making the substitution $u = \sigma_1$, $v = \sigma_{-1}$, so that

$$\phi_i = \begin{cases} u^i & i \geq 0 \\ v^i & i < 0 \end{cases} \tag{A.11}$$

where $u + v = \chi_1$ and $uv = 1$. Then (A.10) becomes

$$S_m = \phi_m + \phi_{m-2} + ... + \phi_{-m} \tag{A.12}.$$

The coefficients of $\{\phi_i\}$ appearing in the decomposition of $\{S_m\}$ are non-negative, since they arise from the decomposition of group characters. Alternatively, as $|| A_{\infty,\infty} || = 2$, we know from Lemma 4.2 that $S_m(\Delta) \geq 0$ for all m, where $\Delta = \Delta(A_{\infty,\infty})$ which gives independent confirmation of this fact. Also by Theorem 4.5, the matrix $R = [R_{ij}]$ satisfying $R\Delta_{\Gamma_\chi} = \Delta_{\Gamma_\sigma} R$ with $R_{ij} \geq 0$, and $R* = *$ is

$$R = [a_0, a_1, ...] = \begin{pmatrix} & . & . & 0 & . & . \\ & . & . & 0 & 1 & . & . \\ & . & 0 & 1 & 0 & . & . \\ & 0 & 1 & 0 & 1 & . & . \\ & 1 & 0 & 1 & 0 & . & . \\ & 0 & 1 & 0 & 1 & . & . \\ & . & 0 & 1 & 0 & . & . \\ & . & . & 0 & 1 & . & . \\ & & . & . & 0 & . & . \end{pmatrix} \tag{A.13}$$

where $a_o^T = (0,...0,1,0....) = (\delta_{io})$, $a_m = S_m(\Delta(A_{\infty,\infty}))a_o$.

We now define $P_i \in \mathbb{Z}[\tau]$, by

$$P_i(\tau) = x^{-i} S_i(x) \tag{A.14}$$

if $\tau = x^{-2}$, $i = 0,1,2,....$ so that P_i are the Jones polynomials [J]. Polynomials $P_{(i,n)}$ associated with the vertices of $\hat{A}_\infty$, the Bratteli diagram for $A(A_\infty) \cong (\otimes M_2)^{SU(2)}$, are then defined by

$$P_{(i,n)}(\tau) = \tau^{(n-i)/2} P_i(\tau) = x^{-n} S_i(x) \tag{A.15}.$$

Similarly define $Q_{(i,n)}$ for $(i,n) \in \hat{A}^{(0)}_{\infty,\infty}$ by

$$Q_{(i,n)}(t) = x^{-n} \phi_i \tag{A.16}.$$

If $\frac{u}{x} = t$, $\frac{v}{x} = 1 - t$, and $i > 0$ then

$$Q_{(i,n)}(t) = u^i/x^n = (u^i/x^i)(1/x^{n-i}) = t^i\,\tau^{(n-i)/2}.$$

But $\tau = x^{-2} = (u/x)(v/x) = t(1-t)$, and so

$$Q_{(i,n)}(t) = t^{(n+i)/2}(1-t)^{(n-i)/2} \in \mathbb{Z}[t], \tag{A.17}$$

which are the polynomials that appear in [R].

The dimension groups of the AF algebras $(\otimes M_2)^{\mathbb{T}}$ and $(\otimes M_2)^{SU(2)}$ were first characterized by Renault [R] and Wassermann [W] respectively. The methods of section 3 recover their results:

$$K_0((\otimes M_2)^{SU(2)}) \cong \mathbb{Z}[\tau] = \mathrm{lin}_{\mathbb{Z}}\,\{P_{(i,n)}(\tau)\} \tag{A.18}$$

with positive cone $K_0((\otimes M_2)^{SU(2)})_+$ identified with

$$\{0\} \cup \{f \in \mathbb{Z}[\tau] : f(\lambda) > 0,\ \lambda \in (0,\tfrac{1}{4}]\} = \mathrm{lin}_{\mathbb{N}}\{P_{(i,n)}(\tau)\},$$

$$K_0((\otimes M_2)^{\mathbb{T}}) \cong \mathbb{Z}[t] = \mathrm{lin}_{\mathbb{Z}}\,\{Q_{(i,n)}(t)\} \tag{A.19}$$

with positive cone $K_0((\otimes M_2)^{\mathbb{T}})$ identified with

$$\{0\} \cup \{f \in \mathbb{Z}[t] : f(t) > 0,\ \ t \in (0,1)\}.$$

Now since $\tau = t(1-t)$, there is an inclusion $K_0((\otimes M_2)^{SU(2)}) \to K_0((\otimes M_2)^{\mathbb{T}})$ given by $f(\tau) \to g(t)$, where $g(t) = f(t(1-t))$. This map is clearly positive, i.e. if $f \in \mathbb{Z}[\tau]$, $f(\lambda) > 0$ for $\lambda \in (0,\frac{1}{4}]$, then $g(\eta) = f(\eta(1-\eta)) > 0$ for $\eta \in (0,1)$. This implies that we can express the polynomials $P_{(i,n)}$ as non-negative integer linear combinations of the polynomials $Q_{(j,m)}$. In fact using (A.12) we have

$$P_n(\tau) = \sum_{k=0}^{n} t^k (1-t)^{n-k} \tag{A.20}$$

so that

$$P_{(i,n)}(\tau) = \tau^{(n-i)/2} P_i(\tau) = \sum_{k=0}^{n} t^{(n+2k-i)/2}(1-t)^{(n-(2k-i))/2}$$

$$= \sum_{k=0}^{i} Q_{(2k-i,n)}(t) \tag{A.21}.$$

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